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Discussion in Mar., Apr., 1947		May 1, 1947
de Vries, Karl. Strength of Beams as Determined by Lateral Buckling	Sept., 1946	
Discussion in Dec., 1946, Feb., Mar., Apr., 1947		May 1, 1947
Ward, C. N., and Hunt, Henry J. Correction of Tailwater Erosion at Prairie Du Sac Dam	Oct., 1946	May 1, 1947
Benscoter, Stanley U. Matrix Analysis of Continuous Beams	Oct., 1946	
Discussion in Feb., Apr., 1947		May 1, 1947
Machis, Alfred. Experimental Observations on Grouting Sands and Gravels	Nov., 1946	
Discussion in Feb., Mar., 1947		May 1, 1947
Symposium: Cleaning and Grouting of Limestone Foundations, Tennessee Valley Authority	Dec., 1946	
Discussion in Mar., 1947		May 1, 1947
Matthes, Gerard A. Mississippi River Cutoffs	Jan., 1947	
Discussion in Mar., 1947		June 1, 1947
Tan, Ek-Khoo. Stability of Soil Slopes	Jan., 1947	June 1, 1947
Muldrow, W. C. Forecasting Productivity of Irrigable Lands	Feb., 1947	July 1, 1947
Freudenthal, Alfred M. Reflections on Standard Specifications for Structural Design	Feb., 1947	July 1, 1947
Blaisdell, Fred W. Development and Hydraulic Design, Saint Anthony Falls Stilling Basin	Feb., 1947	July 1, 1947
DuVal, Miles P. The Marine Operating Problems, Panama Canal, and the Solution	Feb., 1947	
Discussion in Apr., 1947		Uncertain
Claybourn, J. G. Sea Level Plan for Panama Canal	Feb., 1947	
Discussion in Apr., 1947		Uncertain
Hrennikoff, Alexander. Theory of Inelastic Bending with Reference to Moment Design	Mar., 1947	Aug. 1, 1947
Hsu Shih-Chang. Beam Deflections by Second and Third Moments	Mar., 1947	Aug. 1, 1947
Ryder, Frederick L. A Rational Explanation of Column Behavior	Mar., 1947	Aug. 1, 1947
Report. Organization, Financing, and Administration of Sanitary Districts: Final Report of the Committee of the Sanitary Engineering Division	Mar., 1947	

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* Publication of closing discussion pending.

VOL. 73 PROCEEDINGS No. 4

CONTENTS FOR APRIL, 1947

PAPERS

PAGE

Continuous Frame Analysis by Elastic Support Action.
By J. Charles Rathbun and C. W. Cunningham 413

Friction Coefficients in a Large Tunnel.
By G. H. Hickox, A. J. Peterka, and R. A. Elder 451

Stability of Thin Cylindrical Shells in Torsion.
By R. G. Sturm 471

REPORT

Water Supply Engineering: Report of the Committee of the Sanitary Engineering Division for the Two Years Ending December 31, 1946 497

DISCUSSIONS

The Marine Operating Problems, Panama Canal, and the Solution.
By the Editor 523

Sea Level Plan for Panama Canal.
By the Editor 524

Future Costs and Their Effects on Engineering Budgets.
By Louis R. Howson 525

Moment-Stiffness Relations in Continuous Frames with Prismatic Members.
By William A. Conwell 528

Critical Stresses in a Circular Ring.
By E. A. Ripperger and N. Davids 531

Express Highway Planning in Metropolitan Areas.
By M. Hirschthal, and Joseph Barnett 533

CONTENTS FOR APRIL, 1947 (*Continued*)

PAGE

Design Live Loads in Buildings.
By John W. Dunham 541

Matrix Analysis of Continuous Beams.
By Rufus Oldenburger, Horace A. Johnson, Lloyd T. Cheney, and Paul W. Norton. 543

Recharge and Depletion of Ground-Water Supplies.
By Raphael G. Kazmann 557

Strength of Beams as Determined by Lateral Buckling.
By H. N. Hill, and H. D. Hussey 559

A list of "Current Papers and Discussions" may be found on the page preceding the table of contents

HINTS TO AUTHORS

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AMERICAN SOCIETY OF CIVIL ENGINEERS
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PAPERS

CONTINUOUS FRAME ANALYSIS BY ELASTIC SUPPORT ACTION

By J. Charles Rathbun,[1] M. ASCE, and C. W. Cunningham,[2] Assoc. M. ASCE

Synopsis

The purpose of this paper is to present a "classical" or "exact" method of analyzing continuous frames which requires the direct solution of very few, if any, simultaneous equations. The greatest objection to methods of slope deflection, least work, and virtual work is the labor involved in the solution of these equations. There is some parallelism in form with the moment distribution method but the fundamental ideas are entirely different. The repeated cycles are entirely avoided. Reference is made to this parallelism to clarify the derivations and to help in visualizing the several steps in the solution of the problems.

The procedure recommended for the solution of problems requires the calculation of certain quantities which are analogous to the stiffness factors, carry-over factors, and fixed-end moments of moment distribution. Equations are developed for these quantities using a beam with elastic supports. This support condition is a result of the elasticity of the members of the frame adjacent to the supports. The equations have been previously developed using the flexibility of the connections as the elasticity.[3] The method is particularly well adapted to the analysis of frames with flexible connections.

The necessary equations are derived, the general method of procedure is outlined, and then the solution of a progressive series of problems is given in detail or outlined so that it can be followed readily.

Introduction

The method is presented as a series of problems beginning with a simple case and then progressing to more complicated cases as the technique of solution

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by September 1, 1947.

[1] Prof., Civ. Eng., College of the City of New York, New York, N. Y.

[2] Asst. Prof., Civ. Eng., College of the City of New York, New York, N. Y.

[3] "Elastic Properties of Riveted Connections," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 101, 1936, pp. 548–552 and 558.

becomes more apparent. A tabular arrangement of required values may be used since the solution follows a fixed pattern.

PART A. DERIVATION OF EQUATIONS

The letter symbols in this paper are defined where they first appear, in the text or by illustration.

The properties of the conjugate beam as presented in standard texts on strength of materials[4,5] are used in the derivations. If the conjugate beam is loaded with the $\frac{M}{E\,I}$-diagram of the given beam, the resulting shear and bending moment at any point are equal, respectively, to the slope and deflection at the corresponding point on the elastic curve of the given beam.

The conjugate beam has the same span length as the given beam. The reactions must be consistent with the $\frac{M}{E\,I}$-loading, and they must also produce shears and moments consistent with the conditions imposed on the given beam. The sign convention will be correct if the following notation is used:

(1) The shear (slope) at any section is positive if the resultant of all forces $\left(\frac{M}{E\,I}\right)$ to the left of the section acts in a positive or upward direction;

(2) The bending moment (deflection) at any section is positive if it causes compression in the upper fibers of the beam;

(3) The $\frac{M}{E\,I}$-diagram has the same sign as the M-diagram and is taken as an upward load if positive; and

(4) The left reaction of the conjugate beam is equal to the slope of the elastic curve at the left support. Similarly, the right reaction is equal to the slope at the right support with its sign changed.

In any continuous frame one or both of the supporting points of a member are elastic by virtue of the flexibility of the other members framing into these points. This elasticity (which is assumed to be the result of flexural deforma-

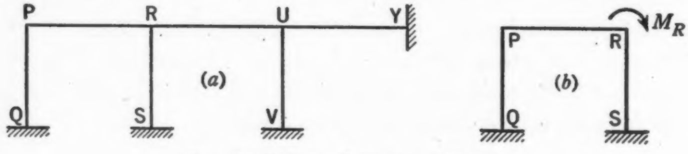


FIG. 1.—TYPICAL CONTINUOUS FRAME

tion only) may be measured by the angular change of a section at the point resulting from a unit moment, assuming that the member under consideration is not acting.

Member RU, Fig. 1(*a*), represents a beam on elastic supports. The elasticity of support U depends on the flexibility of members UV and UY and the

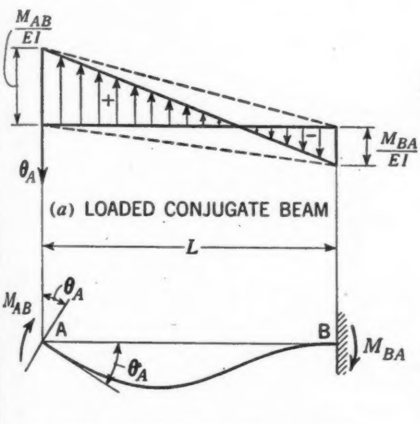
[4] "Resistance of Materials," by Fred B. Seely, John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1935, pp. 298 and 312.

[5] "Elements of Strength of Materials," by S. Timoshenko and G. H. McCullough, D. Van Nostrand Co., Inc., New York, N. Y., 2d Ed., 1940, pp. 167 and 183.

elasticity of support R depends on the flexibility of members RS, RP, and PQ. At each of these supports these flexibilities can be combined into one quantity, called the "elastic factor," Z. The method of computing and using this quantity and others required for the analysis will be shown.

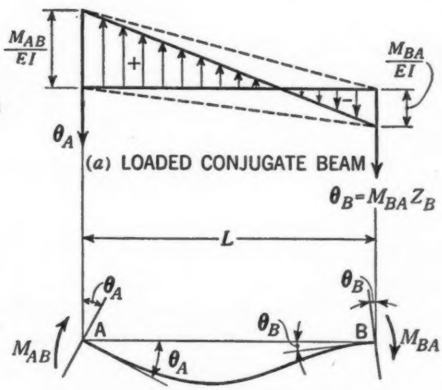
1. The Carry-Over Factor and the Elastic Factor for a Prismatic Beam Having a Fixed Support at One End

Fig. 2(*b*) shows the elastic curve of a given beam AB having a span of length L in which point B is a support assumed to be fixed. Beam UY in Fig. 1(*a*) is



(*b*) LOAD AND DEFLECTION DIAGRAM

Fig. 2.—Fixed Support at One End



(*b*) LOAD AND DEFLECTION DIAGRAM

Fig. 3.—Elastic Support at Both Ends

such a member. The beam is rotated at point A by an applied moment M_{AB} which produces an induced moment M_{BA} at support B. The loaded conjugate beam is shown in Fig. 2(*a*). Since the slope of the given beam AB at point B is zero, the reaction of the conjugate beam B is also zero. For convenience the $\frac{M}{E\,I}$-diagram of Fig. 2(*a*) has been divided into two triangles (one positive and one negative) by the dotted lines.

Taking moments about point A on the conjugate beam: $\frac{M_{AB}}{E\,I} \times \frac{L}{2} \times \frac{L}{3} - \frac{M_{BA}}{E\,I} \times \frac{L}{2} \times \frac{2\,L}{3} = 0$; or

$$M_{BA} = \tfrac{1}{2}\, M_{AB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In Eq. 1, the fraction $\frac{1}{2}$ is the well-known "carry-over factor" for a fixed support at point B. The "elastic factor" is found by computing the reaction θ_A of the conjugate beam. Since θ_B is zero, θ_A is equal to the area of the area of the $\frac{M}{E\,I}$-diagram; or

$$\theta_A = \frac{M_{AB}}{E\,I} \times \frac{L}{2} - \frac{M_{BA}}{E\,I} \times \frac{L}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

from which, by Eq. 1,

$$\theta_A = \frac{M_{AB}\,L}{4\,E\,I} \quad \text{(2b)}$$

In Eq. 2b $\frac{L}{4\,E\,I}$ is the "elastic factor" or angular change at point A due to the application of a unit moment. Representing the elastic factor at point A by Z_A, with the far end, B, fixed:

$$Z_A = \frac{L}{4\,E\,I} \quad \text{(3)}$$

If point B is a hinged support, the $\frac{M}{E\,I}$-diagram is a single triangle since M_{BA} is zero, and

$$\theta_A = \frac{M_{AB}\,L}{3\,E\,I} \quad \text{(4)}$$

Since the elastic factor Z_A is equal to $\frac{\theta_A}{M_{AB}}$, it follows:

$$Z_A = \frac{L}{3\,E\,I} \quad \text{(5)}$$

2. The Carry-Over Factor and the Elastic Factor for a Prismatic Beam Having an Elastic Support at End B, the End Farthest from the Applied Moment

Fig. 3(*b*) shows the elastic curve of a beam AB similar in every respect to that of Fig. 2 except that the slope θ_B at end B is proportional to M_{BA}. From the definition of Z:

$$\theta_B = M_{BA}\,Z_B \quad \text{(6)}$$

Taking moments about end A on the conjugate beam: $-\theta_B\,L - \frac{M_{BA}}{E\,I} \times \frac{L}{2} \times \frac{2\,L}{3} + \frac{M_{AB}}{E\,I} \times \frac{L}{2} \times \frac{L}{3} = 0$. Substituting $M_{BA}\,Z_B$ for θ_B and multiplying by $-\frac{3\,E\,I}{L}$: $3\,E\,I\,M_{BA}\,Z_B + M_{BA}\,L - \frac{M_{AB}\,L}{2} = 0$; $(L + 3\,E\,I\,Z_B)\,M_{BA} = \frac{M_{AB}\,L}{2}$; and

$$M_{BA} = \frac{M_{AB}}{2} \times \frac{L}{L + 3\,E\,I\,Z_B} = \frac{M_{AB}}{2} \times \frac{L}{L_{AB}} \quad \text{(7)}$$

in which $\frac{L}{2\,L_{AB}}$ is the carry-over factor when support B is elastic. The quantity $(L + 3\,E\,I\,Z_B)$ will be called the "elastic length" of the beam with respect to the support B and will be denoted by L_{AB} to simplify the form of the equations involving this expression; thus:

$$L_{AB} = L + 3\,E\,I\,Z_B \quad \text{(8)}$$

The "elastic factor" is obtained by computing the reactions, θ, of the conjugate beam in Fig. 3(a):

$$-\theta_A L + \frac{M_{AB}}{E I} \times \frac{L}{2} \times \frac{2 L}{3} - \frac{M_{BA}}{E I} \times \frac{L}{2} \times \frac{L}{3} = 0 \qquad (9a)$$

and

$$\theta_A = \frac{M_{AB} L}{3 E I} - \frac{M_{BA} L}{6 E I} \qquad (9b)$$

From Eq. 7,

$$\theta_A = \frac{M_{AB} L}{3 E I} - \frac{M_{AB} L^2}{12 E I L_{AB}} \qquad (10a)$$

That is,

$$\theta_A = M_{AB} \left(\frac{4 L L_{AB} - L^2}{12 E I L_{AB}} \right) \qquad (10b)$$

and

$$Z_A = \frac{\theta_A}{M_{AB}} = \frac{4 L L_{AB} - L^2}{12 E I L_{AB}} = \frac{L}{3 E I} \left[\frac{L_{AB} - \frac{L}{4}}{L_{AB}} \right] \qquad (11)$$

Eq. 11 reduces to Eq. 3 when point B is fixed since Z_B is zero and L_{AB} becomes L.

The general use of Eq. 11 will be shown by outlining the procedure for computing the elastic factor for the frame RPQ of Fig. 1(a). The elastic factor Z_{RPQ} depends on the flexibility of the column of PQ which may be obtained from Eq. 3. Therefore, Z_{RPQ} is found by substituting Z_{PQ} from Eq. 3 in Eq. 11. Inasmuch as Z_A in Eq. 11 is a function of Z_B, the latter must be computed first. Thus, one must start with a point such as a fixed or hinged joint whose rigidity is known.

The procedure for computing the elastic factor for part of a frame such as QPRS of Fig. 1(a) will now be given. Fig. 1(b) shows the part QPRS isolated as a free body and subjected to an external positive moment M_R. Since the sum of the moments about a joint is zero, the sum of the moments on joint R from RPQ and RS is equal but opposite in sign to the moment M_R. The elastic factor representing the combined flexibility of RPQ and RS is found by noting that each of the aforementioned moments is proportional to the stiffness factor of the member which is equal to the reciprocal of the elastic factor. The moment relation at joint R is

$$M_R = -(M_{RS} + M_{RP}) \qquad (12)$$

Since $\theta = M Z$,

$$\frac{\theta_R}{Z_R} = -\left(-\frac{\theta_R}{Z_{RS}} - \frac{\theta_R}{Z_{RPQ}} \right) \qquad (13a)$$

and

$$\frac{1}{Z_R} = \frac{1}{Z_{RS}} + \frac{1}{Z_{RPQ}} \qquad (13b)$$

in which Z_R is the angular change at joint R when a unit moment is applied to the part QPRS at that joint.

3. End Moments Due to Transverse Loading for a Prismatic Beam on Elastic Supports

Fig. 4(*b*) represents the elastic curve of a beam AB of span length L, loaded transversely and having two flexible supports A and B. The moment diagram for the given transverse loads acting on a simple beam of length L is shown in Fig. 4(*c*). The moment diagrams due to M_{AB} and M_{BA} are shown in Figs. 4(*d*) and 4(*e*), respectively. The principle of superposition of the simple-beam moment diagrams is used to obtain the moment diagram for the loads and moments acting simultaneously. The resulting curve, divided by $E\,I$, is shown in Fig. 4(*a*).

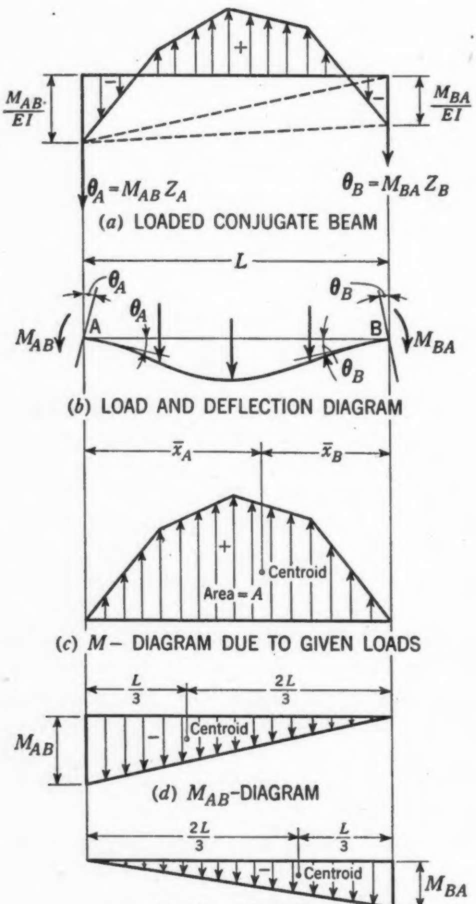


FIG. 4.—BEAM ON ELASTIC SUPPORTS, LOADED TRANSVERSELY

Using the $\frac{M}{E\,I}$-diagram in Fig. 4(*a*) as a load, the reactions, θ_A and θ_B, of the conjugate beam are found by taking moments about supports B and A, respectively. The rotation θ in each equation is then replaced by its corresponding value, $M\,Z$, and the equations are solved simultaneously to obtain formulas for the end moments. Taking moments about end B: $-\theta_A L - \frac{M_{AB}}{E\,I} \times \frac{L}{2} \times \frac{2\,L}{3} - \frac{M_{BA}}{E\,I} \times \frac{L}{2} \times \frac{L}{3} + \frac{A}{E\,I} \times \bar{x}_B = 0$; $3\,E\,I\,\theta_A + M_{AB}\,L + \frac{M_{BA}\,L}{2} - \frac{3\,A\,\bar{x}_B}{L} = 0$; $3\,E\,I\,M_{AB}\,Z_A + M_{AB}\,L + \frac{M_{BA}\,L}{2} - \frac{3\,A\,\bar{x}_B}{L} = 0$; and

$$M_{AB}\,(L + 3\,E\,I\,Z_A) + \frac{M_{BA}\,L}{2} = \frac{3\,A\,\bar{x}_B}{L} \qquad (14a)$$

For brevity, let:

$$L + 3\,E\,I\,Z_A = L_{BA} \qquad (14b)$$

—from which:

$$M_{AB}\,L_{BA} + \frac{M_{BA}\,L}{2} = \frac{3\,A\,\bar{x}_B}{L} \qquad (15a)$$

Similarly, by taking moments about point A:

$$M_{BA}\,L_{AB} + \frac{M_{AB}\,L}{2} = \frac{3\,A\,\bar{x}_A}{L} \qquad (15b)$$

The solution of the simultaneous equations (Eqs. 15*a* and 15*b*) is given by Eqs. 16, as follows:

$$M_{AB} = \frac{6\,A}{L}\left(\frac{2\,L_{AB}\,\bar{x}_B - L\,\bar{x}_A}{4\,L_{AB}\,L_{BA} - L^2}\right) \qquad (16a)$$

and

$$M_{BA} = \frac{6\,A}{L}\left(\frac{2\,L_{BA}\,\bar{x}_A - L\,\bar{x}_B}{4\,L_{AB}\,L_{BA} - L^2}\right) \qquad (16b)$$

If one end, such as B, is a hinged support, $M_{BA} = 0$. The numerator and denominator of Eq. 16*a* are then divided by L_{AB} producing

$$M_{AB} = \frac{6\,A}{L}\left[\frac{2\,\bar{x}_B - \dfrac{L\,\bar{x}_A}{L_{AB}}}{4\,L_{BA} - \dfrac{L^2}{L_{AB}}}\right] \qquad (16c)$$

For a hinge, $Z = \infty$; consequently, $L_{AB} = \infty$ by Eq. 8. The last term in both the numerator and the denominator of Eq. 16*c* becomes zero. This equation then reduces to

$$M_{AB} = \frac{3\,A\,\bar{x}_B}{L_{BA}\,L} \qquad (16d)$$

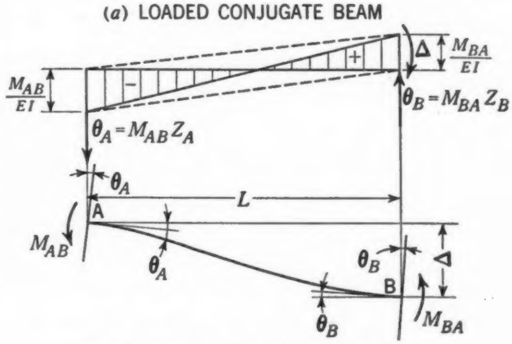
If ends A and B are fixed supports, L_{AB} and L_{BA} of Eqs. 16*a* and 16*b* are replaced by L since Z_A and Z_B are both zero. The resulting equations are the well-known formulas for fixed-end moments in the moment distribution method:

$$M_{AB} = \frac{2\,A}{L^2}\,(2\,\bar{x}_B - \bar{x}_A) \qquad (17a)$$

and

$$M_{BA} = \frac{2\,A}{L^2}\,(2\,\bar{x}_A - \bar{x}_B) \qquad (17b)$$

(*a*) LOADED CONJUGATE BEAM



(*b*) LOAD AND DEFLECTION DIAGRAM

FIG. 5.—UNEQUAL SETTLEMENT OF SUPPORTS

4. END MOMENTS DUE TO THE RELATIVE DISPLACEMENT OF THE ENDS OF A PRISMATIC BEAM ON ELASTIC SUPPORTS

These displacements are perpendicular to the longitudinal axis of the beam. Fig. 5(*b*) shows the elastic curve of a beam AB of span length L when the end B is displaced a distance Δ relative to end A. The induced moments are M_{AB} and M_{BA}. The loaded conjugate beam is shown in Fig. 5(*a*). The displacement of end B, shown as a negative quantity in Fig. 5(*b*), is represented as a

negative moment Δ acting on the conjugate beam at support B. Taking moments about end B on the conjugate beam: $-\theta_A L - \frac{M_{AB}}{E I} \times \frac{L}{2} \times \frac{2L}{3} + \frac{M_{BA}}{E I} \times \frac{L}{2} \times \frac{L}{3} + \Delta = 0$; and

$$M_{AB} L + 3 E I \theta_A - \frac{M_{BA} L}{2} = \frac{3 E I \Delta}{L} \qquad (18)$$

Substituting the value of θ_A and also L_{BA} for $L + 3 E I Z_A$:

$$M_{AB} L_{BA} - \frac{M_{BA} L}{2} = \frac{3 E I \Delta}{L} \qquad (19a)$$

Similarly, taking moments about end A and substituting the corresponding values for θ_B and $L + 3 E I Z_B$ (Eq. 8):

$$M_{BA} L_{AB} - \frac{M_{AB} L}{2} = \frac{3 E I \Delta}{L} \qquad (19b)$$

The solution of the simultaneous equations (Eqs. 19) is given by Eqs. 20, as follows:

$$M_{AB} = \frac{6 E I \Delta}{L} \left(\frac{2 L_{AB} + L}{4 L_{AB} L_{BA} - L^2} \right) = \frac{3 E I \Delta}{2 L} \left[\frac{2 L_{AB} + L}{L_{AB} L_{BA} - \frac{L^2}{4}} \right] \qquad (20a)$$

and

$$M_{BA} = \frac{6 E I \Delta}{L} \left(\frac{2 L_{BA} + L}{4 L_{AB} L_{BA} - L^2} \right) = \frac{3 E I \Delta}{2 L} \left[\frac{2 L_{BA} + L}{L_{AB} L_{BA} - \frac{L^2}{4}} \right] \qquad (20b)$$

If points A and B are fixed supports, L_{AB} and L_{BA} of Eqs. 20 are replaced by L since Z_A and Z_B are both zero. Then these equations reduced to the joint translation equations of moment distribution:

$$M_{AB} = M_{BA} = \frac{6 E I \Delta}{L^2} \qquad (21)$$

5. Flexible Connections

In general, the flexibility of the connections can be taken into consideration by increasing the elastic length of the beams. The "elastic length" of the member used in the computations will then be

$$L' = L + 3 E I Z_J + 3 E I Z_C \qquad (22)$$

in which Z_J refers to the joint and Z_C to the connection. The value of Z_C for the connection must be obtained by experiment or by other methods, as explained elsewhere.[3]

However, Eq. 11 cannot be used for flexible connections because θ_A in this formula is the result of joint rotation only. Therefore, it will be necessary to derive a new expression for the "elastic factor."

Since the rotation at the joint is equal to the rotation of the end of the beam plus the angular change in the connection:

$$\theta_{AJ} = \frac{M_{AB}\,L}{3\,E\,I}\left[\frac{L_{AB} - \frac{L}{4}}{L_{AB}}\right] + \theta_{AC} \qquad (23a)$$

and

$$Z_{AJ} = \frac{\theta_{AJ}}{M_{AB}} = \frac{L}{3EI}\left[\frac{L_{AB} - \frac{L}{4}}{L_{AB}}\right] + Z_{AC} = \frac{L}{3EI}\left[\frac{L_{AB} + 3\,E\,I\,\frac{L_{AB}}{L}\,Z_{AC} - \frac{L}{4}}{L_{AB}}\right]$$

$$= \frac{L\,L_{AB} + 3\,E\,I\,Z_{AC}\,L_{AB} - \frac{L^2}{4}}{3\,E\,I\,L_{AB}} = \frac{(L + 3\,E\,I\,Z_{AC})\,L_{AB} - \frac{L^2}{4}}{3\,E\,I\,L_{AB}} \qquad (23b)$$

For brevity, let:

$$L + 3\,E\,I\,Z_{AC} = L'_{AC} \qquad (24)$$

from which

$$Z_{AJ} = \frac{L'_{AC}\,L_{AB} - \frac{L^2}{4}}{3\,E\,I\,L_{AB}} \qquad (25)$$

If the connection of AB at end B is flexible—

$$L_{AB} = L + 3\,E\,I\,(Z_B + Z_{BC}) \qquad (26)$$

—as stated in Eq. 22.

6. Method of Procedure

If the frame is one in which there is joint translation (sidesway), the solution of the problem may be divided into six distinct steps. In those cases where the solution involves joint translation, a solution is first obtained assuming that this motion is prevented. Later this assumption is corrected by additional steps.

In the derivation of formulas it was emphasized that the calculation of the elastic factors must begin with a member adjacent to a primary support (a joint whose rigidity is known). However, in some structures (the Vierendeel truss being an example), this is not possible. The computations for the elastic factors cannot be begun where a path can be followed to its starting point. This condition can be avoided temporarily by securing certain joints against rotation. These "fixed joints" can later be released and the effect of their adjustment superimposed on the solution already obtained. In many frames the "fixed joints" are not required.

Step (*a*).—Select a minimum number of "fixed joints," dividing the frame into sections such that in any section one can move on the frame from any point in the section to any other by one, and only one, route.

Step (*b*).—Compute the elastic factors Z_A and Z_B for each member of the frame (see Eqs. 3, 5, or 11). The elastic lengths and the carry-over factors are also computed.

Step (*c*).—Compute the moments at the ends of the members connected to each "fixed joint" when the joint is rotated a unit amount. Distribute the resulting moments throughout the frame.

Step (*d*).—Compute and distribute the end moments of the loaded members by Eqs. 16.

Step (*e*).—The temporary "fixed joints" are eliminated by forming simultaneous equations from the resulting steps (*c*) and (*d*). Solve these equations for the rotation correction factors.

Step (*f*).—Eliminate the fixed condition of the joints by substituting by proportion the results of step (*e*) in step (*c*), and combine the results with those of step (*d*). If there is no joint translation (sidesway), the result of step (*f*) is the final solution of the problem.

7. Joint Translation

Each problem contains a definite number of shear requirements that must be satisfied. These result in deformations across the section where they occur. Literal values are assigned to these deformations and an equation of condition is set up for each section. The literal terms of each equation represent the shears across the sections due to the several (unknown) deformations. The solution of the simultaneous equations will give the values of the unknowns. The final moments are found by the principle of superposition, adding the change in the moments due to these deformations to those found by step (*f*). The shear corrections require three additional steps.

Step (*g*).—Translate the joints, successively, along each shear section an equal, arbitrary amount allowing no rotation except that due to the flexibility of the supporting members; and then compute the resulting end moments from Eqs. 20. Distribute these moments in the same manner as those due to a load.

Step (*h*).—Compute the shear on each shear section due to the joint translations of step (*g*). Form equations of condition for the several shear sections noting that these shears are the coefficients of the literal deformations. The solution of these equations gives the amount of deflection necessary to satisfy the shear requirements of the problem.

Step (*i*).—Multiply the moments of step (*g*) by the shear correction factors obtained from the solution of the equations of step (*h*) and add the results to the moments found by step (*f*). The final result is the complete solution of the problem.

The steps in the illustrative problems are lettered to correspond to the foregoing step letters. If a step is not required for a problem, it will be omitted.

A study of Eqs. 3 to 20 will show that E is eliminated from the calculations when the elastic factor from Eq. 3 or Eq. 5 is substituted in the succeeding equations. For convenience in computations involving individual equations, E will be assumed equal to unity. The moment of inertia, I, will be given a proportional value since it is seldom the same for all members of the structure.

In evaluating the formulas, the customary sign convention for moments used in beam analysis has been followed—that is, a moment is positive if it produces compression on the upper fibers of the beam. This notation presents difficulties in frame analysis so, as in moment distribution, the convention used in the problems will be: A positive end moment is one that will tend to turn the end of a member going into a joint in a clockwise direction.

PART B. ILLUSTRATIVE EXAMPLES

EXAMPLE 1 (*a*)

A four-span continuous beam ABCDE (Fig. 6) carries a 4-kip concentrated load at the center of span BC. The left end, A, is fixed and the right end, E, is hinged. The moment of inertia, I, is taken as 15 and is included in the computations for illustrative purposes although the solution is independent of its value. The moments at points A, B, C, and D are the required quantities.

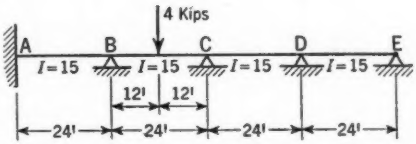


FIG. 6.—FOUR-SPAN CONTINUOUS BEAM WITH ONE CONCENTRATED LOAD

Step (*a*).—Since there are no closed circuits and since the computations for the elastic factors begin with the single members AB and DE, no joints fixed against rotation are needed.

Step (*b*).—To provide a starting point for the computations, assume that points B and C are isolated elastic supports, the elasticity of point B resulting from the flexibility of member BA and that of point C from the flexibility of member CDE. The required elastic factors will be calculated first: For member BA, Eq. 3 gives $Z_{BA} = \dfrac{24}{4 \times 15} = \dfrac{1}{2.500}$; and, for member DE, Eq. 5 gives $Z_{DE} = \dfrac{24}{3 \times 15} = \dfrac{1}{1.875}$. The elastic factor Z_{CDE} for member CDE is calculated from Eq. 11 which involves the elastic length L_{CD} of member CD; thus (see Eq. 8): $L_{CD} = L + 3\,E\,I\,Z_{DE} = 24 + 3 \times 15 \times \dfrac{1}{1.875} = 24 + 24 = 48.00$;

and (see Eq. 11) $Z_{CDE} = \dfrac{24}{3 \times 15}\left[\dfrac{48 - \dfrac{24}{4}}{48}\right] = \dfrac{1}{2.143}$.

The elastic lengths, L_{BC} and L_{CB}, needed in step (*d*) are (see Eq. 8): $L_{BC} = L + 3\,E\,I\,Z_{CDE} = 24 + 3 \times 15 \times \dfrac{1}{2.143} = 45.00$; and $L_{CB} = L + 3\,E\,I\,Z_{BA} = 24 + 3 \times 15 \times \dfrac{1}{2.500} = 42.00$. The required carry-over factors are: Points B to A (Eq. 1), 0.50; points C to D (Eq. 7) $\dfrac{24}{2 \times 48} = 0.25$; and points D to E = 0 (as is known) because (see Eq. 7) $L_{DE} = \infty$.

The elastic length of member DE is infinite since $Z_E = \infty$ due to the hinge.

Step (*d*).—The maximum ordinate of the moment diagram for the 4-kip concentrated load is $\dfrac{P\,L}{4} = \dfrac{4 \times 24}{4} = 24$ ft-kips. Therefore, $A = \frac{1}{2} \times 24 \times 24 = 288$; and $\bar{x}_B = \bar{x}_C = 12$. Then, by Eq. 16*a*,

$$M_{BC} = \frac{6 \times 288}{24}\left(\frac{2 \times 45 \times 12 - 24 \times 12}{4 \times 45 \times 42 - 24 \times 24}\right) = 8.165;$$

and, by Eq. 16*b*,

$$M_{CB} = \frac{6 \times 288}{24}\left(\frac{2 \times 42 \times 12 - 24 \times 12}{4 \times 45 \times 42 - 24 \times 24}\right) = 7.423.$$

Moments M_{BC} and M_{CB} for this problem are tabulated and distributed in Table 1, with the sign notation as mentioned in "Part A: 6. Method of Procedure."

TABLE 1.—MOMENTS AT THE SUPPORTS OF THE CONTINUOUS BEAM ABCDE, FIG. 6

AB	BA	BC	CB	CD	DC	DE	ED
−4.082	−8.165	+8.165	−7.423	+7.423	+1.856	−1.856	0

EXAMPLE 1(*b*)

The same four-span continuous beam as in Example 1(*a*) (Fig. 6) carries a 4-kip load at the center of span BC and a load of 8 kips uniformly distributed over span CD.

Step (*b*).—Compute the following values in addition to those already found in Example 1(*a*): For member CBA, $Z_{CBA} = \frac{24}{3 \times 15}\left[\frac{42 - \frac{24}{4}}{42}\right] = \frac{1}{2.188}$ (see Eq. 11); and L_{DC} (see Eq. 8) $= 24 + 3 \times 15 \times \frac{1}{2.188} = 24 + 20.57 = 44.57$.

The carry-over factor from points C to B is $\frac{24}{2 \times 42} = 0.286$.

Step (*d*).—Find the end moments due to the 8-kip uniform load. The maximum ordinate of the moment-curve diagram (a second degree parabola) is $\frac{W\,L}{8} = \frac{8 \times 24}{8} = 24$ ft-kips. The area under a second degree parabola is equal to two thirds of the area of the circumscribed rectangle. Therefore, $A = \frac{2}{3} \times 24 \times 24 = 384$ and $\bar{x}_C = \bar{x}_D = 12$.

Then, by Eq. 16*a*,

$$M_{CD} = \frac{6 \times 384}{24}\left(\frac{2 \times 48 \times 12 - 24 \times 12}{4 \times 48 \times 44.57 - 24 \times 24}\right) = 10.392;$$

and, by Eq. 16*b*,

$$M_{DC} = \frac{6 \times 384}{24}\left(\frac{2 \times 44.57 \times 12 - 24 \times 12}{4 \times 48 \times 44.57 - 24 \times 24}\right) = 9.402.$$

Moments M_{BC} and M_{CB} from Example 1(*a*), and moments M_{CD} and M_{DC} in this example, are tabulated and distributed in Table 2, the last line of which gives the moments at the several joints, created by the two loads.

TABLE 2.—TABULATION AND DISTRIBUTION OF MOMENTS, EXAMPLE 1(*b*)

Line	Description	AB	BA	BC	CB	CD	DC	DE	ED
1	Moments created by elasticity of supports	....	....	+8.165	− 7.423	+10.392	−9.402	....	....
2	Joint distribution.....	....	−8.165	....	−10.392	+ 7.423	....	+9.402	....
3	Carry-over[a]..........	−2.598	+2.969	−2.969	....	....	+1.856	−1.856	0
4	Final moments.......	−2.598	−5.196	+5.196	−17.815	+17.815	−7.546	+7.546	0

[a] For member AB, 5.196 × 0.50 = 2.598; for member BA, 10.392 × 0.286 = 2.969; and, for member DC, 7.423 × 0.25 = 1.856.

EXAMPLE 2

A rigid frame (Fig. 7) carries a 4-kip concentrated load at the center of the beam AC. The moments at points A, B, C, and D are required.

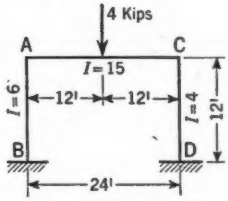


FIG. 7.—RIGID SYMMETRICAL FRAME

Step (b).—It will be assumed that points A and C are isolated elastic supports. The elasticity of joint A is a function of the flexibility of member AB and the elasticity of joint C is a function of the flexibility of member CD. Eq. 3 yields for member AB, $Z_{AB} = \frac{12}{4 \times 6} = \frac{1}{2}$; and, for member CD, $Z_{CD} = \frac{12}{4 \times 4} = \frac{3}{4}$. The elastic lengths (Eq. 8) are: $L_{AC} = 24 + 3 \times 15 \times \frac{3}{4} = 24 + 33.75 = 57.75$; and $L_{CA} = 24 + 3 \times 15 \times \frac{1}{2} = 24 + 22.50 = 46.50$. By Eq. 11: For member ACD—$Z_{ACD} = \frac{24}{3 \times 15}\left[\frac{57.75 - \frac{24}{4}}{57.75}\right] = \frac{1}{2.092}$, and, for member CAB, $Z_{CAB} = \frac{24}{3 \times 15}\left[\frac{46.50 - \frac{24}{4}}{46.50}\right] = \frac{1}{2.153}$.

Step (g).—The elastic lengths required in step (g) are:

$$L_{AB} = L + 3\,E\,I\,Z_B \quad = 12 + 3 \times 6 \times 0 \quad = 12.00$$

$$L_{BA} = L + 3\,E\,I\,Z_{ACD} = 12 + 3 \times 6 \times \frac{1}{2.092} = 20.60$$

$$L_{CD} = L + 3\,E\,I\,Z_D \quad = 12 + 3 \times 4 \times 0 \quad = 12.00$$

$$L_{DC} = L + 3\,E\,I\,Z_{CAB} = 12 + 3 \times 4 \times \frac{1}{2.153} = 17.56$$

The required carry-over factors are: By Eq. 1—points A to B = 0.50 and points C to D = 0.50; and, by Eq. 7—points A to C $= \frac{24}{2 \times 57.75} = 0.208$ and points C to A $= \frac{24}{2 \times 46.50} = 0.258$.

Step (d).—From Example 1(a), $A = 288$ and $\bar{x}_A = \bar{x}_C = 12$. Then, by Eqs. 16: $M_{AC} = \frac{6 \times 288}{24}\left(\frac{2 \times 57.75 \times 12 - 24 \times 12}{4 \times 57.75 \times 46.50 - 24 \times 24}\right) = 7.777$; and $M_{CA} = \frac{6 \times 288}{24}\left(\frac{2 \times 46.50 \times 12 - 24 \times 12}{4 \times 57.75 \times 46.50 - 24 \times 24}\right) = 5.865$. These moments are tabulated and distributed in the first line of Table 3(b).

Step (g).—According to statics the shear equation

$$\frac{M_{AB} + M_{BA} + M_{CD} + M_{DC}}{12}$$

should equal zero; but the actual shear across members AB and CD is

$$\frac{-\,7.777 - 3.888 + 5.865 + 2.932}{12} = \frac{-\,2.868}{12};$$

and, therefore, as should be expected, joints A and C must move laterally until the induced moments produce a shear that will counteract the effect of the shear $\frac{-2.868}{12}$. Joint A will be translated a distance Δ, which for convenience is taken as $\frac{12}{E}$, laterally to the right of joint B. However, the direction of the relative movement is entirely arbitrary. In a like manner joint C will be

TABLE 3.—TABULATION AND DISTRIBUTION OF MOMENTS, EXAMPLE 2

Line	Description	BA	AB	AC	CA	CD	DC
	(a) STEP (g)						
1	Moments created by elasticity of supports.	+2.267	+1.534	....	....	+1.236	+1.618
2	Joint distribution	....	....	−1.534	−1.236	....	....
3	Carry-over[a]	+0.160	+0.319	−0.319	−0.319	+0.319	+0.160
4	Final moments	+2.427	+1.853	−1.853	−1.555	+1.555	+1.778
	(b) STEPS (d) AND (i)						
5	Moments created by the load, assuming that sidesway is prevented (step (d))	−3.888	−7.777	+7.777	−5.865	+5.865	+2.932
6	Correction for sidesway (line 4 × 0.376)	+0.914	+0.698	−0.698	−0.586	+0.586	+0.670
7	Final moments (step (i))	−2.974	−7.079	+7.079	−6.451	+6.451	+3.602

[a] For members BA and DC, 0.319 × 0.50 = 0.160; for member AC, 1.236 × 0.258 = 0.319; and, for member CA, 1.534 × 0.208 = 0.319.

translated the same distance to the right of point D. The induced moments (see Eq. 17) are:

$$M_{AB} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 12 + 12}{4 \times 12 \times 20.62 - 12 \times 12}\right) = 1.534$$

$$M_{BA} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 20.62 + 12}{4 \times 12 \times 20.62 - 12 \times 12}\right) = 2.267$$

$$M_{CD} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 12 + 12}{4 \times 12 \times 17.60 - 12 \times 12}\right) = 1.236$$

$$M_{DC} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 17.60 + 12}{4 \times 12 \times 17.60 - 12 \times 12}\right) = 1.618$$

These moments are tabulated and distributed in Table 3(*a*). The shear across members AB and CD due to these moments is:

$$\frac{+1.853 + 2.427 + 1.555 + 1.778}{12} = \frac{+7.613}{12}.$$

Step (*h*).—The shear correction factor is: $\frac{2.868}{12} \div \frac{7.613}{12} = \frac{2.868}{7.613} = 0.376.$ The sign of the shear from step (*g*) has been changed to produce reverse balancing moments.

Step (*i*).—Line 7, Table 3(*b*) gives the final moments for the frame including the effect of joint translation.

Example 3

A four-span rigid frame is loaded as shown in Fig. 8. The end moments of all members are required.

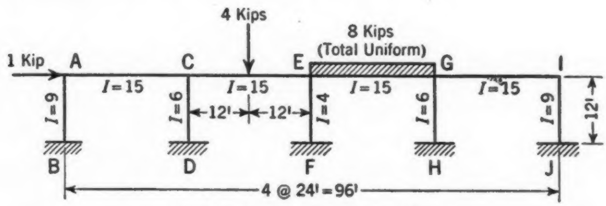


Fig. 8.—Four-Span Rigid Frame

Step (*b*).—The necessary elastic lengths and elastic factors are found, first, by progressing from joint I to joint A and then from joint A to joint I. In the computations in Table 4 it will be found convenient to express the elastic factor

TABLE 4.—Computations for Step (*b*), Example 3

Member	Elastic Length			Stiffness Factor			
	Symbol	Computation	Feet	Symbol	Computation	Feet	Eq.
IJ......	L_{IJ}	....	12.00	$1/Z_{IJ}$	$4 \times 9/12$	3.000	3
GI......	L_{GI}	$24 + \frac{3 \times 15}{3.000}$	39.00	$1/Z_{GI}$	$\frac{3 \times 15 \times 39}{24(39 - 6)}$	2.216	11
GH......	L_{GH}	....	12.00	$1/Z_{GH}$	$4 \times 6/12$	2.000	3
EG......	L_{EG}	$24 + \frac{3 \times 15}{2.216 + 2.000}$	34.67	$1/Z_{EG}$	$\frac{3 \times 15 \times 34.67}{24(34.67 - 6)}$	2.267	11
EF......	L_{EF}	....	12.00	$1/Z_{EF}$	$4 \times 4/12$	1.333	3
CE......	L_{CE}	$24 + \frac{3 \times 15}{2.267 + 1.333}$	36.50	$1/Z_{CE}$	$\frac{3 \times 15 \times 36.50}{24(36.50 - 6)}$	2.244	11
CD......	L_{CD}	....	12.00	$1/Z_{CD}$	$4 \times 6/12$	2.000	3
AC......	L_{AC}	$24 + \frac{3 \times 15}{2.244 + 2.000}$	34.60	$1/Z_{AC}$	$\frac{3 \times 15 \times 34.60}{24(34.60 - 6)}$	2.268	11
BA......	L_{BA}	$12 + \frac{3 \times 9}{2.268}$	23.90	....	....	....	....

in its reciprocal form, which is the "stiffness factor." The remaining values are found by symmetry; thus:

Member	L	$1/Z$
AB...............	12.00	3.000
CA...............	39.00	2.216
EC...............	34.67	2.267
GE...............	36.50	2.244
IG...............	34.60	2.268
JI...............	23.90	....

The following elastic lengths are required for the sidesway calculations and are based on stiffness factors in Table 4: For member DC,

$$L_{DC} = 12 + \frac{3 \times 6}{2.216 + 2.244} = 16.04;$$

for member FE, $L_{FE} = 12 + \frac{3 \times 4}{2 \times 2.267} = 14.65$; and, for member HG, $L_{HG} = 16.04$.

The carry-over factors are found by Eq. 7; thus: From point A to point C, $\frac{24}{2 \times 34.60} = 0.347$; from point C to point E, $\frac{24}{2 \times 36.45} = 0.329$; from point E to point G, $\frac{24}{2 \times 34.70} = 0.346$; and, from point G to point I, $\frac{24}{2 \times 39.00} = 0.308$. Carry-over factors, from I to G, G to E, E to C, and C to A, are found by symmetry; and the carry-over factors from the column tops to the bases are all 0.500.

Step (d).—The values of A, $\bar{x}_A$, and $\bar{x}_B$, for the 4-kip concentrated load and the 8-kip uniform load, are taken from Examples 1(*a*) and 1(*b*). The end moments from each of these loads are computed by Eqs. 16*a* and 16*b*:

$$M_{CE} = \frac{6 \times 288}{24}\left(\frac{2 \times 36.50 \times 12 - 24 \times 12}{4 \times 36.50 \times 34.67 - 24 \times 24}\right) = 9.436$$

$$M_{EC} = \frac{6 \times 288}{24}\left(\frac{2 \times 34.67 \times 12 - 24 \times 12}{4 \times 36.50 \times 34.67 - 24 \times 24}\right) = 8.734$$

$$M_{EG} = \frac{6 \times 384}{24}\left(\frac{2 \times 34.67 \times 12 - 24 \times 12}{4 \times 34.67 \times 36.50 - 24 \times 24}\right) = 11.645$$

$$M_{GE} = \frac{6 \times 384}{24}\left(\frac{2 \times 36.50 \times 12 - 24 \times 12}{4 \times 34.67 \times 36.50 - 24 \times 24}\right) = 12.582$$

To facilitate the distribution of moments, the stiffness factors of the members framing into joints C, E, and G are listed as follows:

Member	Stiffness factor
Joint C—	
CA	2.216
CD	2.000
CE	2.244
Joint E—	
EC	2.267
EF	1.333
EG	2.267
Joint G—	
GE	2.244
GH	2.000
GI	2.216

The end moment from any member is distributed to the other members at the joint in direct proportion to its stiffness. The moments caused by the loads, tabulated in line 6, Table 5(*b*), are distributed first to the right (line 7) and then to the left (line 8) by the proper stiffness and carry-over factors. Since the moments in line 6 are the result of elastic support action, there is no carry-over for $M_{EC} = -8.734$ to joint C or for $M_{EG} = -11.645$ to joint G, the moment $M_{EC} = -8.734$ being distributed to members EF and EG as $+3.234$ and $+5.500$, respectively. The part of $+5.500$ carried over to joint G is $+1.903$. This is combined with -12.582 to produce a net moment $M_{GE} = -10.679$ which is distributed to members GH and GI. Line 9, Table 5(*b*), gives the final moments for this step.

Step (*g*).—If the system is in equilibrium, the sum of the end moments for the columns should equal 1 kip $\times$ 12 ft or 12.000 ft-kips. However, the actual sum of these moments listed in line 9, Table 5(*b*), is $+0.157$, showing that an additional $+11.843$ must be furnished. To effect this adjustment the joints A, C, E, G, and I, Fig. 8, will be translated a convenient distance $\Delta = \frac{12}{E}$ laterally to the right. Using Eqs. 20*a* and 20*b*, the induced moments are:

$$M_{AB} = \frac{6 \times 9 \times 12}{12} \times \left(\frac{2 \times 12 + 12}{4 \times 12 \times 23.90 - 12 \times 12}\right) = 1.938$$

$$M_{BA} = \frac{6 \times 9 \times 12}{12} \times \left(\frac{2 \times 23.90 + 12}{4 \times 12 \times 23.90 - 12 \times 12}\right) = 3.219$$

$$M_{CD} = \frac{6 \times 6 \times 12}{12} \times \left(\frac{2 \times 12 + 12}{4 \times 12 \times 16.04 - 12 \times 12}\right) = 2.071$$

$$M_{DC} = \frac{6 \times 6 \times 12}{12} \times \left(\frac{2 \times 16.04 + 12}{4 \times 12 \times 16.04 - 12 \times 12}\right) = 2.536$$

$$M_{EF} = \frac{6 \times 4 \times 12}{12} \times \left(\frac{2 \times 12 + 12}{4 \times 12 \times 14.65 - 12 \times 12}\right) = 1.546$$

$$M_{FE} = \frac{6 \times 4 \times 12}{12} \times \left(\frac{2 \times 14.65 + 12}{4 \times 12 \times 14.65 - 12 \times 12}\right) = 1.773$$

$$M_{GH} = \quad \cdots\cdot \quad = 2.071$$

$$M_{HG} = \quad \cdots\cdot \quad = 2.536$$

$$M_{IJ} = \quad \cdots\cdot \quad = 1.938$$

$$M_{JI} = \quad \cdots\cdot \quad = 3.219$$

These moments are tabulated and distributed in Table 5(*a*). The sum of the end moments for the columns is 25.204.

Step (*h*).—The shear correction factor is: $\frac{11.843}{25.204} = 0.470$.

Step (*i*).—Multiply the moments in line 5, Table 5(*a*), by 0.470 and tabulate the results in line 10, Table 5(*b*). The addition of lines 9 and 10 gives the final moments for the frame including the effect of joint translation.

EXAMPLE 4

A single-panel, open-web (Vierendeel) truss is loaded as shown in Fig. 9. The end moments of all members are required.

TABLE 5.—TABULATION AND DIS

Line	Description	Columns: BA	DC	FE	HG	JI	Joint A: AB	AC
	(a) Step (g)							
1	Moments resulting from elasticity of supports	+3.219	+2.536	+1.773	+2.536	+3.219	+1.938	....
2	Joint distribution	....	....	....	....	....	....	−1.938
3	Carry-over to the right	....	+0.158	+0.042	+0.052	+0.141	....	....
4	Carry-over to the left	+0.141	+0.052	+0.042	+0.158	....	+0.281	−0.281
5	Final moments	+3.360	+2.746	+1.857	+2.746	+3.360	+2.219	−2.219
	(b) Step (f) and Step (i)							
6	Moments resulting from elasticity of supports, for 4-kip and 8-kip loads	....	....	....	....	....	....	....
7	Joint distribution and carry-over (right)	....	....	+1.617	+2.533	−0.863	....	....
8	Joint distribution and carry-over (left)	+0.558	−1.636	−2.156	....	....	+1.116	−1.116
9	Final moments with no sidesway	+0.558	−1.636	−0.539	+2.533	−0.863	+1.116	−1.116
10	Correction for sidesway (line 5 × 0.470)	+1.579	+1.290	+0.873	+1.290	+1.579	+1.043	−1.043
11	Final moments including sidesway	+2.137	−0.346	+0.334	+3.823	+0.716	+2.159	−2.159

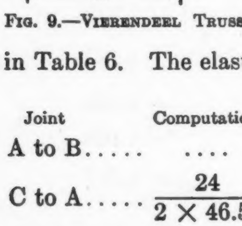


FIG. 9.—VIERENDEEL TRUSS

Step (a).—To compute the elastic factors and the elastic lengths for the several members it is necessary to start with a single member having a fixed or hinged joint at one end. The idea of the "fixed joint" is now introduced for the purpose of establishing, temporarily, a joint of known rigidity. Joint B is assumed to be temporarily secured against rotation.

Step (b).—Computations for step (b) are as shown in Table 6. The elastic and carry-over factors are as follows:

Joint	Computation	Carry-over factor	Joint	Computation	Carry-over factor
A to B	....	0.500	D to B	....	0.500
C to A	$\frac{24}{2 \times 46.50}$	0.258	C to D	$\frac{12}{2 \times 18.00}$	0.333
D to C	$\frac{12}{2 \times 17.57}$	0.341	A to C	$\frac{24}{2 \times 61.50}$	0.195
B to D	$\frac{24}{2 \times 53.85}$	0.223	B to A	$\frac{12}{2 \times 20.66}$	0.290

Step (c).—Eq. 11 is used to obtain a general expression for the moments induced in the ends of the members framing into a joint when the joint is

rotated a unit amount. If θ_A is equal to unity, Eq. 11 becomes:

$$M_{AB} = \frac{1}{Z_{AB}} \qquad (27)$$

TRIBUTION OF MOMENTS, EXAMPLE 3

Joint C			Joint E			Joint G			Joint I		Line
CA	CD	CE	EC	EF	EG	GE	GH	GI	IG	IJ	
(a) Step (g)											
....	+2.071	....	....	+1.546	....	....	+2.071	....	....	+1.938	1
-1.029	....	-1.042	-0.773	....	-0.773	-1.042	....	-1.029	-1.938	....	2
-0.672	+0.317	+0.355	-0.226	+0.084	+0.142	-0.218	+0.104	+0.115	-0.281	+0.281	3
+0.115	+0.103	-0.218	+0.142	+0.084	-0.226	+0.355	+0.317	-0.672	....	....	4
-1.586	+2.491	-0.905	-0.857	+1.714	-0.857	-0.905	+2.492	-1.586	-2.219	+2.219	5
(b) Step (d) and Step (i)											
....	....	+9.436	-8.734	....	+11.645	-12.582	....	....	....	....	6
....	....	....	....	+3.234	+5.500	+1.903	+5.066	+5.613	+1.727	-1.727	7
-3.626	-3.273	-2.538	-7.333	-4.312	....	....	....	....	....	....	8
-3.626	-3.273	+6.898	-16.067	-1.078	+17.145	-10.679	+5.066	+5.613	+1.727	-1.727	9
-0.745	+1.171	-0.425	-0.403	+0.806	-0.403	-0.425	+1.171	-0.746	-1.043	+1.043	10
-4.371	-2.102	+6.473	-16.470	-0.272	+16.742	-11.104	+6.237	+4.867	+0.684	-0.684	11

The "fixed joint" B is rotated a unit amount producing the following moments which are assumed positive for convenience: $M_{BA} = 1.755$ and $M_{BD} = 1.688$. The moment M_{BA} (= 1.755) is tabulated and distributed in line 1, Table 7(*a*), assuming that member BD is secured against rotation during the operation; $M_{BD} = 1.688$ is tabulated and distributed in line 2; and line 3 contains the final moments for this step.

The external moment required to rotate joint B a unit amount is found as follows: Let M_B be the external moment. Then $M_B + M_{BA} + M_{BD} = 0$; and $M_B = -(M_{BA} + M_{BD}) = -(1.738 + 1.671) = -3.409$. The quantity M_B is required for the calculation of the rotation correction factor of steps (*e*) and (*g*).

Step (*d*).—From Example 1(*a*), $A = 288$ and $\bar{x}_A = \bar{x}_C = 12$. Then

$$M_{AC} = \frac{6 \times 288}{24}\left(\frac{2 \times 61.50 \times 12 - 24 \times 12}{4 \times 61.50 \times 46.50 - 24 \times 24}\right) = 7.874;$$

and

$$M_{CA} = \frac{6 \times 288}{24}\left(\frac{2 \times 46.50 \times 12 - 24 \times 12}{4 \times 61.50 \times 46.50 - 24 \times 24}\right) = 5.488.$$

Moments M_{AC} and M_{CA} are tabulated and distributed in line 10, Table 7(*c*).

Step (*e*).—The "fixed joint" B is eliminated in the following manner: The external moment required to hold joint B fixed against rotation, as assumed in the computations for step (*d*), is equal but opposite in sign to the sum of the

internal moments, M_{BA} and M_{BD} in line 10, Table 7(*c*). This external moment is $-(-3.937 - 0.914) = +4.851$; and, to release the joint, an additional external moment of -4.851 is also applied at joint B. The rotation correction factor is $\frac{-4.851}{-3.409} = +1.423$.

Step (*f*).—To find the moments in the members due to the external moment of -4.852 applied at joint B, multiply the moments in line 3, Table 7(*a*), by $+1.423$ and enter the product in line 11, Table 7(*c*). Line 12, Table 7(*c*), gives the final moments if sidesway is prevented.

TABLE 6.—COMPUTATIONS FOR STEP (*b*), EXAMPLE 4

Member	Elastic Length			Elastic Factor		
	Symbol	Computation	Feet	Symbol	Computation	Feet
AB.......	L_{AB}	....	12.00	Z_{AB}	$\frac{12}{4 \times 6}$	$\frac{1}{2.000}$
CA.......	L_{CA}	$24 + \frac{3 \times 15}{2.000}$	46.50	Z_{CA}	$\frac{24}{3 \times 15}\left(\frac{46.50 - 6}{46.50}\right)$	$\frac{1}{2.153}$
DC.......	L_{DC}	$12 + \frac{3 \times 4}{2.153}$	17.57	Z_{DC}	$\frac{12}{3 \times 4}\left(\frac{17.57 - 3}{17.57}\right)$	$\frac{1}{1.206}$
BD.......	L_{BD}	$24 + \frac{3 \times 12}{1.206}$	53.85	Z_{BD}	$\frac{24}{3 \times 12}\left(\frac{53.85 - 6}{53.85}\right)$	$\frac{1}{1.688}$
DB.......	L_{BD}	....	24.00	Z_{DB}	$\frac{24}{4 \times 12}$	$\frac{1}{2.000}$
CD.......	L_{CD}	$12 + \frac{3 \times 4}{2.000}$	18.00	Z_{CD}	$\frac{12}{3 \times 4}\left(\frac{18.00 - 3}{18.00}\right)$	$\frac{1}{2.000}$
AC.......	L_{AC}	$24 + \frac{3 \times 15}{1.200}$	61.50	Z_{AC}	$\frac{24}{3 \times 15}\left(\frac{61.50 - 6}{61.50}\right)$	$\frac{1}{2.078}$
BA.......	L_{BA}	$12 + \frac{3 \times 6}{2.078}$	20.66	Z_{BA}	$\frac{12}{3 \times 6}\left(\frac{20.66 - 3}{20.67}\right)$	$\frac{1}{1.755}$

Step (*g*).—For equilibrium the sum of the end moments for members AB and CD should equal 12×0 or 0 since there are no lateral loads. The actual sum of these moments from line 12, Table 7(*c*), is -1.872, showing that an additional $+1.872$ must be furnished by the lateral movement of joints A and C relative to joints B and D. For the purpose of finding the lateral deformation required for equilibrium, joints A and C are translated a convenient distance $\Delta = 12/E$ laterally to the right. The induced moments are:

$$M_{AB} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 12 + 12}{4 \times 12 \times 20.66 - 12 \times 12}\right) = 1.528$$

$$M_{BA} = \frac{6 \times 6 \times 12}{12}\left(\frac{2 \times 20.66 + 12}{4 \times 12 \times 20.66 - 12 \times 12}\right) = 2.264$$

$$M_{CD} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 18 + 12}{4 \times 18 \times 17.57 - 12 \times 12}\right) = 1.027$$

$$M_{DC} = \frac{6 \times 4 \times 12}{12}\left(\frac{2 \times 17.57 + 12}{4 \times 18 \times 17.57 - 12 \times 12}\right) = 1.009$$

These moments are entered in line 4 and distributed in lines 5 and 6, Table 7(*b*), assuming joint B is fixed against rotation. Line 7 gives the moments before releasing joint B. The releasing moment is + 1.843 or 2.397 − 0.554. The rotation correction factor is $\frac{+\ 1.843}{-\ 3.409}$ or − 0.541.

TABLE 7.—Tabulation and Distribution of Moments, Example 4

Line	Description	BA	AB	AC	CA	CD	DC	DB	BD
	(*a*) Step (*c*) (Distribution After Rotating Joint B)								
1	Rotation of member BA only..	+1.755	+0.509	−0.509	−0.099	+0.099	+0.033	−0.033	−0.017
2	Rotation of member BD only..	−0.017	−0.033	−0.033	+0.128	−0.128	−0.376	+0.376	+1.688
3	Final moments, step (*c*).......	+1.738	+0.476	−0.476	+0.029	−0.029	−0.343	+0.343	+1.671
	(*b*) Steps (*g*) and (*h*)								
4	Moments caused by the elasticity of the supports, assuming joint B fixed against rotation....................	+2.264	+1.528	....	....	+1.027	+1.009	....	....
5	Distribution to the right......	....	....	−1.528	−0.298	+0.298	+0.099	−1.108	−0.554
6	Distribution to the left........	+0.133	+0.265	−0.265	−1.027	....	....	....	....
7	Final moments resulting from translation, assuming joint B fixed against rotation.......	+2.397	+1.793	−1.793	−1.325	+1.325	+1.108	−1.108	−0.554
8	Moments after release of Joint B (line 3 × − 0.541).......	−0.939	−0.257	+0.257	−0.016	+0.016	+0.185	−0.185	−0.904
9	Final moments resulting from translation...............	+1.458	+1.536	−1.536	−1.341	+1.341	+1.293	−1.293	−1.458
	(*c*) Steps (*d*), (*f*), (*g*), and (*i*)								
10	Joint B fixed against rotation and no sidesway............	−3.937	−7.874	+7.874	−5.488	+5.488	+1.829	−1.829	−0.914
11	Joint B released (line 3 × + 1.423).............	+2.473	+0.678	−0.678	+0.041	−0.041	−0.488	+0.488	+2.378
12	Final moments without sidesway.......................	−1.464	−7.196	+7.196	−5.447	+5.447	+1.341	−1.341	+1.464
13	Correction for sidesway (line 9 × + 0.333).............	+0.485	+0.511	−0.511	−0.446	+0.446	+0.430	−0.430	−0.485
14	Final moments..............	−0.979	−6.685	+6.685	−5.893	+5.893	+1.771	−1.771	+0.979

Multiply the moments in line 3, Table 7(*a*), by − 0.541 and enter the product in line 8, Table 7(*b*). Line 9 gives the moments due to the lateral displacement of $12/E$ including the effect of releasing joint B.

Step (*h*).—The sum of the end moments for members AB and CD, from line 9, Table 7(*b*), is + 5.628. The shear correction factor is $\frac{+\ 1.872}{+\ 5.628} = +\ 0.333$.

Step (*i*).—Multiply the moment in line 9, Table 7(*b*), by + 0.333 and tabulate the result in line 13, Table 7(*c*). The addition of lines 12 and 13 gives the final moments for the frame including the effect of joint translation laterally (see line 14, Table 7(*c*)).

Example 5

A double-panel, open-web truss is loaded as shown in Fig. 10. The end moments of all members are required.

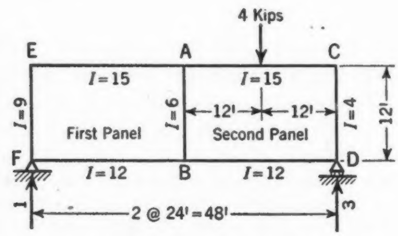


FIG. 10.—DOUBLE-PANEL OPEN-WEB TRUSS

Step (a).—Joint B is assumed to be temporarily secured against rotation.

TABLE 8.—COMPUTATIONS FOR STEP (*b*), EXAMPLE 5

Member	Elastic Length			Stiffness Factor		
	Symbol	Computation	Feet	Symbol	Computation	Feet
DB.......	L_{DB}	....	24.00	$1/Z_{DB}$	....	2.000
CD.......	L_{CD}	....	18.00	$1/Z_{CD}$	....	1.200
AC.......	L_{AC}	....	61.50	$1/Z_{AC}$	....	2.078
AB.......	L_{AB}	....	12.00	$1/Z_{AB}$	....	2.000
EA.......	L_{EA}	$24 + \frac{3 \times 15}{2.000 + 2.078}$	35.04	$1/Z_{EA}$	$\frac{3 \times 15}{24}\left(\frac{35.04}{35.04 - 6}\right)$	2.262
FE.......	L_{FE}	$12 + \frac{3 \times 9}{2.262}$	23.93	$1/Z_{FE}$	$\frac{3 \times 9}{12}\left(\frac{23.93}{23.93 - 3}\right)$	2.572
BF.......	L_{BF}	$24 + \frac{3 \times 12}{2.572}$	37.99	$1/Z_{BF}$	$\frac{3 \times 12}{24}\left(\frac{37.99}{37.99 - 6}\right)$	1.781
FB.......	L_{FB}	....	24.00	$1/Z_{FB}$	....	2.000
EF.......	L_{EF}	$12 + \frac{3 \times 9}{2.000}$	25.50	$1/Z_{EF}$	$\frac{3 \times 9}{12}\left(\frac{25.50}{25.50 - 3}\right)$	2.550
AE.......	L_{AE}	$24 + \frac{3 \times 15}{2.550}$	41.65	$1/Z_{AE}$	$\frac{3 \times 15}{24}\left(\frac{41.65}{41.65 - 6}\right)$	2.191
BA.......	L_{BA}	$24 + \frac{3 \times 6}{2.191 + 2.078}$	16.22	$1/Z_{BA}$	$\frac{3 \times 6}{12}\left(\frac{16.22}{16.22 - 3}\right)$	1.840
CA.......	L_{CA}	$24 + \frac{3 \times 15}{2.000 + 2.191}$	34.74	$1/Z_{CA}$	$\frac{3 \times 15}{24}\left(\frac{34.74}{34.74 - 6}\right)$	2.266
DC.......	L_{DC}	$12 + \frac{3 \times 4}{2.266}$	17.29	$1/Z_{DC}$	$\frac{3 \times 4}{12}\left(\frac{17.29}{17.29 - 3}\right)$	1.210
BD.......	L_{BD}	$24 + \frac{3 \times 12}{1.210}$	53.76	$1/Z_{BD}$	$\frac{3 \times 12}{24}\left(\frac{53.76}{53.76 - 6}\right)$	1.688

Step (b).—In Table 8, the detailed computations are omitted for quantities that have been computed in Example 4. The elastic and carry-over factors are:

Joint	Computation	Carry-over factor	Joint	Computation	Carry-over factor
D to B.....	....	0.500	F to B.....	....	0.500
C to D.....	....	0.333	E to F.....	$\frac{12}{2 \times 25.50}$	0.235
A to C.....	....	0.195	A to E.....	$\frac{24}{2 \times 41.65}$	0.288
A to B.....	....	0.500	B to A.....	$\frac{12}{2 \times 16.22}$	0.370
E to A.....	$\frac{24}{2 \times 35.04}$	0.343	C to A.....	$\frac{24}{2 \times 34.74}$	0.346
F to E.....	$\frac{12}{2 \times 23.93}$	0.251	D to C.....	$\frac{12}{2 \times 17.29}$	0.347
B to F.....	$\frac{24}{2 \times 37.99}$	0.316	B to D.....	$\frac{24}{2 \times 53.76}$	0.223

Step (*c*).—The "fixed joint" B is rotated a unit amount producing the following moments: $M_{BF} = 1.781$, $M_{BA} = 1.840$, and $M_{BD} = 1.688$. These moments are tabulated and distributed in Table 9(*a*). In line 1, the moment due to the rotation of member BA is distributed as far as members AE and AC. In line 2 the moment due to the rotation of member BF is distributed to member AC where it is combined with that due to the rotation of BA. The resultant moment is then distributed on the remainder of the line. A similar condition exists in line 3 where the distribution is begun with the rotation of member BD. The final moments for this step are given in line 4, Table 9(*a*). The external moment M_B is equal to $-(1.767 + 1.816 + 1.676) = -5.259$.

Step (*d*).—From Example 1(*a*) $A = 288$ and $\bar{x}_A = \bar{x}_C = 12$. Then,

$$M_{AC} = \frac{6 \times 288}{24}\left(\frac{2 \times 61.50 \times 12 - 24 \times 12}{4 \times 61.50 \times 34.74 - 24 \times 24}\right) = 10.733;$$

and

$$M_{CA} = \frac{6 \times 288}{24}\left(\frac{2 \times 34.74 \times 12 - 24 \times 12}{4 \times 61.50 \times 34.74 - 24 \times 24}\right) = 4.930.$$

These moments are tabulated and distributed in line 17, Table 9(*d*).

Step (*e*).—The external moment necessary to release joint B is the sum of the internal moments at point B which are listed in line 17, Table 9(*d*). This releasing moment is -3.573. The rotation correction factor is $\frac{-3.573}{-5.259} = +0.679$.

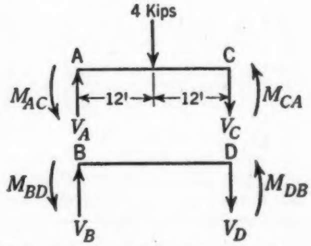


FIG. 11.—FREE-BODY DIAGRAM, SECOND PANEL OF FIG. 10

Step (*f*).—Multiply the moments in line 4, Table 9(*a*), by $+0.679$ to obtain the moments due to -3.573 applied externally at point B and enter the product in line 18, Table 9(*d*). Line 19, Table 9(*d*), gives the final moments if vertical shear deformation is prevented.

Step (*g*).—In the first panel of Fig. 10 the sum of the end moments for members EA and FB should equal 1 kip $\times$ 24 ft for equilibrium, but the actual

sum from line 19, Table 9(*d*), is − 6.428. In the second panel the sum of the end moments for members AC and BD should equal − 1 kip × 24 ft, but the actual sum is + 4.548.

The proof of the statement that the sum of the moments should be − 24 is

TABLE 9.—TABULATION AND DIS-

Line	Description	JOINT B	JOINT F		JOINT E	
		BF	FB	FE	EF	EA
	(a) STEP					
	Distribution After the Rotation of:					
1	Member BA	....	....	....	....	....
2	Members BF and BA	+1.781	+0.563	−0.563	−0.141	+0.141
3	Members BD and BA	−0.014	−0.025	+0.025	+0.107	−0.107
4	Final moments	+1.767	+0.538	−0.538	−0.034	+0.034
	(b) STEP (g) (TRANSLATION					
5	Moments (elasticity of supports)[a]	+2.344	+1.688	....	....	+1.609
6	Distribution of moments, to the right	....	....	−1.688	−0.423	+0.423
7	Distribution of moments, to the left	+0.189	+0.379	−0.379	−1.609	....
8	Translation; joint B fixed against rotation[b]	+2.533	+2.067	−2.067	−2.032	+2.032
9	Joint B released[c] (line 4 × −0.383)	−0.677	−0.206	+0.206	+0.013	−0.013
10	Final moments, translation of member EF	+1.856	+1.861	−1.861	−2.019	+2.019
	(c) STEP (g) (TRANSLATION					
11	Moments (elasticity of supports)[a]	....	....	....	....	....
12	Distribution of moments, to the right	....	....	....	....	....
13	Distribution of moments, to the left	+0.032	+0.064	−0.064	−0.271	+0.271
14	Translation; joint B fixed against rotation[b]	+0.032	+0.064	−0.064	−0.271	+0.271
15	Joint B released[c] (line 4 × +0.339)	+0.598	+0.182	−0.182	−0.011	+0.011
16	Final moments, translation of member CD	+0.630	+0.246	−0.246	−0.282	+0.282
	(d) STEPS (d),					
17	Moments created by load[d] (joint B fixed)	−0.190	−0.380	+0.380	+1.617	−1.617
18	Joint B released[c] (line 4 × +0.679)	+1.200	+0.365	−0.365	−0.023	+0.023
19	Final moments, vertical movement prevented	+1.010	−0.015	+0.015	+1.594	−1.594
20	Member EF[c] (line 10 × +2.980)	+5.532	+5.545	−5.545	−6.018	+6.018
21	Member CD[c] (line 16 × +3.555)	+2.240	+0.873	−0.873	−1.003	+1.003
22	Final moments	+8.782	+6.403	−6.403	−5.427	+5.427

[a] Moments resulting from the elasticity of supports. [b] Final moments resulting from translation, assuming joint B fixed against rotation and vertical movement. [c] Correction for the translation of the member indicated.

given as follows: Fig. 11 shows a free-body sketch of members AC and BD. Since they do not enter into the computations, the direct stresses are omitted. In Fig. 11, $V_A = \dfrac{M_{AC} + M_{CA}}{24} + \dfrac{4 \times 12}{24}$; and $V_B = \dfrac{M_{BD} + M_{DB}}{24}$.

The sum of the shears $V_A + V_B$ is equal to 1 from the first panel and therefore: $\frac{M_{AC} + M_{CA} + M_{BD} + M_{DB}}{24} + 2 = 1$; and $M_{AC} + M_{CA} + M_{BD} + M_{DB} = -1 \times 24$.

TRIBUTION OF MOMENTS, EXAMPLE 5

JOINT A			JOINT C		JOINT D		JOINT B		Line
AE	AB	AC	CA	CD	DC	DB	BD	BA	
(c)									
−0.350	+0.681	−0.331	....	....	....	....	....	+1.840	1
+0.048	−0.024	−0.024	−0.070	+0.070	+0.023	−0.023	−0.012	−0.012	2
−0.021	−0.024	+0.045	+0.131	−0.131	−0.377	+0.377	+1.688	−0.012	3
−0.323	+0.633	−0.310	+0.061	−0.061	−0.354	+0.354	+1.676	+1.816	4
OF MEMBER EF)									
+1.836	....	....	....	....	....	....	....	....	5
+0.145	−0.972	−1.009	−0.197	+0.197	+0.065	−0.065	−0.033	....	6
....	....	....	....	....	....	....	....	−0.486	7
+1.981	−0.972	−1.009	−0.197	+0.197	+0.065	−0.065	−0.033	−0.486	8
+0.124	−0.243	+0.119	−0.024	+0.024	+0.136	−0.136	−0.641	−0.696	9
+2.105	−1.215	−0.890	−0.221	+0.221	+0.201	−0.201	−0.674	−1.182	10
OF MEMBER CD)									
....	....	−1.660	−1.056	....	....	−1.131	−2.066	....	11
....	....	....	....	+1.056	+0.352	−0.352	−0.176	....	12
+0.938	+0.857	−0.136	−0.392	+0.392	+1.131	....	....	+0.428	13
+0.938	+0.857	−1.796	−1.448	+1.448	+1.483	−1.483	−2.242	+0.428	14
−0.109	+0.214	−0.105	+0.021	−0.021	−0.120	+0.120	+0.569	+0.615	15
+0.829	+1.071	−1.901	−1.427	+1.427	+1.363	−1.363	−1.673	+1.043	16
(f), AND (i)									
−5.610	−5.123	+10.733	−4.930	+4.930	+1.643	−1.643	−0.822	−2.561	17
−0.219	+0.430	−0.211	+0.042	−0.042	−0.240	+0.240	+1.139	+1.234	18
−5.829	−4.693	+10.522	−4.888	+4.888	+1.403	−1.403	+0.317	−1.327	19
+6.271	−3.618	−2.653	−0.657	+0.657	+0.599	−0.599	−2.011	−3.521	20
+2.948	+3.809	−6.757	−5.074	+5.074	+4.845	−4.845	−5.950	+3.710	21
+3.390	−4.502	+1.112	−10.619	+10.619	+6.847	−6.847	−7.644	−1.138	22

that joint B is fixed against rotation. [c] Moments after releasing joint B. [d] Moments created by load, assuming

Since the frame is not in equilibrium vertically, deformation sufficient to satisfy the shear requirements must take place across each panel. To compute these deformations, first translate member EF a convenient distance $\Delta = 24/E$ vertically upward relative to member AB. The induced moments are:

$$M_{EA} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 35.04 + 24}{4 \times 35.04 \times 41.65 - 24 \times 24}\right) = 1.609$$

$$M_{AE} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 41.65 + 24}{4 \times 35.04 \times 41.65 - 24 \times 24}\right) = 1.836$$

$$M_{FB} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 24.00 + 24}{4 \times 24.00 \times 37.99 - 24 \times 24}\right) = 1.688$$

$$M_{BF} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 37.99 + 24}{4 \times 24.00 \times 37.99 - 24 \times 24}\right) = 2.344$$

These moments are entered in line 5 and distributed in lines 6 and 7, Table 9(*b*), assuming joint B fixed against rotation. Line 8 gives the moments before releasing joint B. The releasing amount is + 2.014.

The rotation correction factor is $\frac{+\,2.014}{-\,5.259} = -\,0.383$. Multiply the moments in line 4, Table 9(*a*), by − 0.383 and tabulate the product in line 9, Table 9(*b*). Line 10 yields the moments due to the translation of member EF including the effect of releasing the joint.

Next, to compute the deformations, translate member CD a convenient distance $\Delta = 24/E$ vertically upward relative to member AB. The induced moments are:

$$M_{CA} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 34.74 + 24}{4 \times 34.74 \times 61.50 - 24 \times 24}\right) = 1.056$$

$$M_{AC} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 61.50 + 24}{4 \times 34.74 \times 61.50 - 24 \times 24}\right) = 1.660$$

$$M_{DB} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 24.00 + 24}{4 \times 24.00 \times 53.76 - 24 \times 24}\right) = 1.131$$

$$M_{BD} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 53.76 + 24}{4 \times 24.00 \times 53.76 - 24 \times 24}\right) = 2.066$$

These moments are entered in line 11 and distributed in lines 12 and 13, Table 9(*c*). Line 14 gives the moments before releasing joint B. The releasing moment is − 1.782. The rotation correction factor is − 1.782/− 5.259 = + 0.339. Multiply the moments in line 4, Table 9(*a*), by + 0.339 and list the product, line 15, Table 9(*c*). Line 16 gives the moments due to the translation of member CD including the effect of releasing the joint.

Step (*h*).—Let X be the shear correction factor (required deformation across the panel) for the first panel; and let Y be the shear correction factor (required deformation across the panel) for the second panel. The sum of the end moments from the tables for the following members is:

Reference	Members EA and FB	Members AC and BD
Line 10, Table 9(*b*)	+ 7.841	− 1.987
Line 16, Table 9(*c*)	+ 1.987	− 6.364
Line 19, Table 9(*d*)	− 6.428	+ 4.547

From the foregoing, the shear equations can be written as follows: For the first panel—

$$7.841\ X + 1.987\ Y - 6.428 = 24.00 \quad \text{(28a)}$$

and, for the second panel—

$$-\ 1.987\ X - 6.364\ Y + 4.547 = -\ 24.00 \quad \text{(28b)}$$

Solving Eqs. 28 simultaneously, $X = 2.980$ and $Y = 3.555$.

Step (*i*).—Multiply the moments in line 10, Table 9(*b*), by + 2.980 and those in line 16, Table 9(*c*), by + 3.555. Tabulate the results in lines 20 and 21, Table 9(*d*). The addition of lines 19, 20, and 21 gives the final moments including the effect of joint translation (see line 22).

EXAMPLE 6

A four-panel, open-web truss is loaded as shown in Fig. 12. The end moments of all members are required.

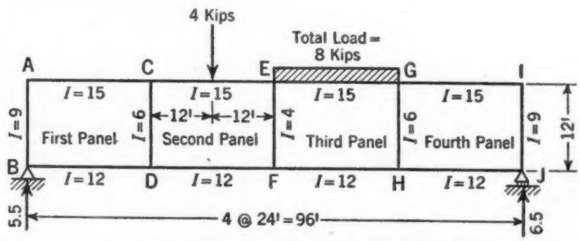


FIG. 12.—FOUR-PANEL OPEN-WEB TRUSS

Step (*a*).—Joints D and H are assumed to be temporarily secured against rotation.

Step (*b*).—Computations for step (*b*) are arranged in Table 10, the elastic lengths and stiffness factors for JH through DB, HG, and HF being determined by symmetry. The carry-over factors are:

Joint	Computation	Carry-over factor	Joint	Computation	Carry-over factor
B to D....	....	0.500	G to E....	$\frac{24}{2 \times 36.797}$	0.326
A to B....	$\frac{12}{2 \times 25.500}$	0.235	G to H....	....	0.500
C to A....	$\frac{24}{2 \times 41.647}$	0.288	I to G....	$\frac{24}{2 \times 34.612}$	0.347
C to D....	....	0.500	J to I....	$\frac{12}{2 \times 23.904}$	0.251
E to C....	$\frac{24}{2 \times 34.738}$	0.345	H to J....	$\frac{24}{2 \times 37.992}$	0.316
F to D....	....	0.500	D to C....	$\frac{12}{2 \times 16.062}$	0.374
F to H....	....	0.500	F to E....	$\frac{12}{2 \times 14.647}$	0.410
E to F....	$\frac{12}{2 \times 15.000}$	0.400	D to F....	$\frac{24}{2 \times 35.051}$	0.342

TABLE 10.—COMPUTATIONS FOR STEP (*b*), EXAMPLE 6

Member	Elastic Length			Stiffness Factor		
	Symbol	Computation	Feet	Symbol	Computation	Feet
BD.......	L_{BD}	....	24.000	$1/Z_{BD}$	$\frac{4\times12}{24}$	2.0000
AB.......	L_{AB}	$12+\frac{3\times9}{2.0000}$	25.500	$1/Z_{AB}$	$\frac{3\times9}{12}\left(\frac{25.500}{25.500-3}\right)$	2.5500
CA.......	L_{CA}	$24+\frac{3\times15}{2.5500}$	41.647	$1/Z_{CA}$	$\frac{3\times15}{24}\left(\frac{41.647}{41.647-6}\right)$	2.1906
CD.......	L_{CD}	....	12.000	$1/Z_{CD}$	$\frac{4\times6}{12}$	2.0000
EC.......	L_{EC}	$24+\frac{3\times15}{2.1906+2.0000}$	34.738	$1/Z_{EC}$	$\frac{3\times15}{24}\left(\frac{34.738}{34.738-6}\right)$	2.2665
FD.......	L_{FD}	....	24.000	$1/Z_{FD}$	$\frac{4\times12}{24}$	2.0000
FH.......	L_{FH}	....	24.000	$1/Z_{FH}$	$\frac{4\times12}{24}$	2.0000
EF.......	L_{EF}	$12+\frac{3\times4}{2.0000+2.0000}$	15.000	$1/Z_{EF}$	$\frac{3\times4}{12}\left(\frac{15.000}{15.000-3}\right)$	1.2500
GE.......	L_{GE}	$24+\frac{3\times15}{1.2500+2.2665}$	36.797	$1/Z_{GE}$	$\frac{3\times15}{24}\left(\frac{36.797}{36.797-6}\right)$	2.2403
GH.......	L_{GH}	....	12.000	$1/Z_{GH}$	....	2.0000
IG.......	L_{IG}	$24+\frac{3\times15}{2.2403+2.0000}$	34.612	$1/Z_{IG}$	$\frac{3\times15}{24}\left(\frac{34.612}{34.612-6}\right)$	2.2682
JI.......	L_{JI}	$12+\frac{3\times9}{2.2682}$	23.904	$1/Z_{JI}$	$\frac{3\times9}{12}\left(\frac{23.904}{23.904-3}\right)$	2.5729
HJ.......	L_{HJ}	$24+\frac{3\times12}{2.5729}$	37.992	$1/Z_{HJ}$	$\frac{3\times12}{24}\left(\frac{37.992}{37.992-6}\right)$	1.7813
JH.......	L_{JH}	....	24.000	$1/Z_{JH}$	....	2.0000
IJ.......	L_{IJ}	....	25.500	$1/Z_{IJ}$	....	2.5500
GI.......	L_{GI}	....	41.647	$1/Z_{GI}$	....	2.1906
EG.......	L_{EG}	....	34.738	$1/Z_{EG}$	....	2.2665
CE.......	L_{CE}	....	36.797	$1/Z_{CE}$	....	2.2403
AC.......	L_{AC}	....	34.612	$1/Z_{AC}$	....	2.2682
BA.......	L_{BA}	....	23.904	$1/Z_{BA}$	....	2.5729
DB.......	L_{DB}	....	37.992	$1/Z_{DB}$	....	1.7813
DC.......	L_{DC}	$12+\frac{3\times6}{2.1906+2.2403}$	16.062	$1/Z_{DC}$	$\frac{3\times6}{12}\left(\frac{16.062}{16.062-3}\right)$	1.8445
HG.......	L_{HG}	....	16.062	$1/Z_{HG}$	....	1.8445
FE.......	L_{FE}	$12+\frac{3\times4}{2\times2.2665}$	14.647	$1/Z_{FE}$	$\frac{3\times4}{12}\left(\frac{14.647}{14.647-3}\right)$	1.2576
DF.......	L_{DF}	$24+\frac{3\times12}{1.2576+2.0000}$	35.051	$1/Z_{DF}$	$\frac{3\times12}{24}\left(\frac{35.051}{35.051-6}\right)$	1.8098
HF.......	L_{HF}	....	35.051	$1/Z_{HF}$	....	1.8098

By symmetry:

Joint	Factor	Joint	Factor	Joint	Factor
				A to C....	0.347
J to H....	0.500	H to G....	0.374	B to A....	0.251
I to J	0.235	E to G....	0.345	D to B....	0.316
G to I	0.288	C to E....	0.326	H to F....	0.342

Step (*c*).—The "fixed joint" D is rotated a unit amount producing the following moments: $M_{DB} = 1.7813$, $M_{DC} = 1.8445$, and $M_{DF} = 1.8098$. These moments are tabulated and distributed as shown in Table 11 under "(*a*) Unit

TABLE 11.—Moments Due to Unit Joint Rotation—Steps (*c*), (*f*), and (*g*), Example 7

Line	Distribution after rotation of:	(*a*) Unit Rotation of Joint D							
		BA	BD	DB	DC	DF	FD	FE	FH
		(*b*) Unit Rotation of Joint H							
		JI	JH	HJ	HG	HF	FH	FE	FD
1	Member DC...........	+0.0231	−0.0231	−0.0115	+1.8445	−0.0040	−0.0081	+0.0162	−0.0081
2	Member DF...........	+0.0006	−0.0006	−0.0003	−0.0040	+1.8098	+0.6196	−0.2392	−0.3804
3	Member DB...........	−0.5626	+0.5626	+1.7813	−0.0115	−0.0003	−0.0006	+0.0012	−0.0006
4	Final moments.........	−0.5389	+0.5389	+1.7695	+1.8290	+1.8055	+0.6109	−0.2218	−0.3891

TABLE 11.—(*Continued*)

Line	(*a*) Unit Rotation of Joint D								
	HF	HG	HJ	JH	JI	AB	AC	CA	CD
	(*b*) Unit Rotation of Joint H								
	DF	DC	DB	BD	BA	IJ	IG	GI	GH
1	−0.0040	−0.0060	−0.0005	−0.0009	+0.0009	+0.0982	−0.0982	−0.3406	+0.6890
2	−0.1902	−0.0040	−0.0003	−0.0006	+0.0006	+0.0026	−0.0026	−0.0089	−0.0080
3	−0.0003	−0.0005	0	−0.0001	+0.0001	−0.1412	+0.1412	+0.0490	−0.0231
4	−0.1945	−0.0105	−0.0008	−0.0016	+0.0016	−0.0404	+0.0404	−0.3005	+0.6579

TABLE 11.—(*Continued*)

Line	(*a*) Unit Rotation of Joint D								
	CE	EC	EF	EG	GE	GH	GI	IG	IJ
	(*b*) Unit Rotation of Joint H								
	GE	EG	EF	EC	CE	CD	CA	AC	AB
1	−0.3484	−0.1136	+0.0404	+0.0732	+0.0253	−0.0121	−0.0132	−0.0038	+0.0038
2	+0.0169	+0.0490	−0.0980	+0.0490	+0.0169	−0.0080	−0.0089	−0.0026	+0.0026
3	−0.0259	−0.0084	+0.0030	+0.0054	+0.0019	−0.0009	−0.0010	−0.0003	+0.0003
4	−0.3574	−0.0730	−0.0546	+0.1276	+0.0441	−0.0210	−0.0231	−0.0067	+0.0067

Rotation of Joint D." Joint H is assumed fixed against rotation during this operation.

Next the "fixed joint" H is rotated a unit amount producing the following moments: $M_{HJ} = 1.7813$, $M_{HG} = 1.8445$, and $M_{HF} = 1.8098$. From symmetry it is evident that the values given in Table 11 under "(*b*) Unit Rotation of Joint H" are the result of distributing these moments throughout the frame. The final moments for this step are given in line 4, Table 11. To compute the external moments M_D and M_H resulting from a unit rotation of joints D and H proceed as in Table 12.

TABLE 12.—COMPUTATION OF EXTERNAL MOMENTS, STEP (*c*), EXAMPLE 6

Symbol	Joint rotated	Computation	External moment
M_D	D	$-(1.7695 + 1.8290 + 1.8055)$	-5.4040
M_H	D	$-(0.1945 - 0.0105 - 0.0008)$	$+0.2058$
M_D	H	....	$+0.2058$
M_H	H	....	-5.4040

Step (*d*).—The values of A and $\bar{x}$ for the 4-kip concentrated load and the 8-kip uniform load are taken from Examples 1(*a*) and 1(*b*). Then:

$$M_{CE} = \frac{6 \times 288}{24} \left(\frac{2 \times 36.797 \times 12 - 24 \times 12}{4 \times 36.797 \times 34.738 - 24 \times 24} \right) = 9.444$$

$$M_{EC} = \frac{6 \times 288}{24} \left(\frac{2 \times 34.738 \times 12 - 24 \times 12}{4 \times 36.797 \times 34.738 - 24 \times 24} \right) = 8.660$$

$$M_{EG} = \frac{6 \times 384}{24} \left(\frac{2 \times 34.738 \times 12 - 24 \times 12}{4 \times 34.738 \times 36.797 - 24 \times 24} \right) = 11.547$$

$$M_{GE} = \frac{6 \times 384}{24} \left(\frac{2 \times 36.797 \times 12 - 24 \times 12}{4 \times 34.738 \times 36.797 - 24 \times 24} \right) = 12.592$$

These moments are entered in line 15 and distributed in line 16, Table 13(*c*), assuming that joints D and H are fixed against rotation and that vertical movement of all joints is prevented. Note that -4.105 listed in line 15, Table 13(*c*), under member EF is the proportional part of 11.547 which is distributed to member EF. Line 17 gives the moments before releasing joints D and H.

Step (*e*).—The external moment necessary to release joints D and H is the sum of the internal moments at each of these joints, as listed in line 17, Table 13(*c*); thus: The releasing moment for $D = -0.122 - 1.640 + 0.103 = -1.659$; and the releasing moment for $H = +0.103 + 2.545 + 0.189 = +2.836$. The external moments at each of these joints due to a unit rotation of each joint have been tabulated at the end of step (*c*). Similar to step (*h*), Example 5, let X_L be the rotation correction factor for joint D, or the amount that joint D must rotate to release both joints, assumed fixed during the load computations for step (*d*); and let Y_L be a similar rotation correction factor for joint H.

TABLE 13.—TABULATION AND DISTRIBUTION OF MOMENTS, EXAMPLE 6

Line	Description	Joint B		Joint D			Joint F		
		BA	BD	DB	DC	DF	FD	FE	FH
	(a) STEPS *(g)* AND *(i)*								
1	Moments (elasticity of supports)[a]	....	+1.6879	+2.3440	....	....	....	....	....
2	Distribution to the right	−1.6879	....	....	−0.4733	−0.0123	−0.0246	+0.0492	−0.0246
3	Distribution to the left	−0.3801	+0.3801	+0.1900	....	....	....	....	....
4	Translation; joints D and H fixed[b]	−2.0680	+2.0680	+2.5340	−0.4733	−0.0123	−0.0246	+0.0492	−0.0246
	Moments After Releasing:								
5	Joint D	+0.0245	−0.0245	−0.6713	−0.6939	−0.6850	−0.2318	+0.0842	+0.1467
6	Joint H	0	0	0	+0.0001	+0.0017	+0.0033	+0.0019	−0.0052
7	Final moments, translation of AB	−1.8635	+1.8635	+1.8627	−1.1671	−0.6956	−0.2531	+0.1353	+0.1178
	(b) STEPS *(g)* AND *(i)*								
8	Moments (elasticity of supports)[a]	....	....	....	....	+2.4294	+1.8588	....	....
9	Distribution of EC to EG and EF; and EF to joints D and H	....	....	....	....	+0.0659	+0.1318	−0.2636	+0.1318
10	Distribution of FD, CE, and EG	+0.0704	−0.0704	−0.0352	−0.4741	....	....	−0.7176	−1.1412
11	Translation; joints D and H fixed[b]	+0.0704	−0.0704	−0.0352	−0.4741	+2.4953	+1.9906	−0.9812	−1.0094
	Moments After Releasing:								
12	Joint D	+0.1968	−0.1968	−0.6461	−0.6678	−0.6593	−0.2231	+0.0810	+0.1421
13	Joint H	+0.0010	−0.0010	0	−0.0007	−0.0121	−0.0242	−0.0138	+0.0380
14	Final moments, translation of CD	+0.2682	−0.2682	−0.6813	−1.1426	+1.8239	+1.7433	−0.9140	−0.8293
	(c) STEPS *(d)*, *(e)*, *(f)*, *(h)*, AND *(i)*								
15	Load moments[c]	....	....	....	....	....	....	....	....
16	Distribution	+0.244	−0.244	−0.122	−1.640	+0.103	+0.205	−0.410	+0.205
17	Final moments; joints D and H fixed[c]	+0.244	−0.244	−0.122	−1.640	+0.103	+0.205	−0.410	+0.205
18	After releasing joint D	−0.155	+0.155	+0.509	+0.526	+0.519	+0.176	−0.064	−0.112
19	After releasing joint H	−0.001	+0.001	0	+0.005	+0.100	+0.200	+0.114	−0.314
20	Final moments; no vertical movement	+0.088	−0.088	+0.387	−1.109	+0.722	+0.581	−0.360	−0.221
	Shear Correction for the:								
21	First panel	−41.118	+41.118	+41.099	−25.751	−15.348	−5.583	+2.984	+2.599
22	Second panel	+4.188	−4.188	−10.640	−17.843	+28.483	+27.224	−14.274	−12.950
23	Third panel	+0.527	−0.527	−1.313	−2.292	+3.605	+9.308	+10.260	−19.568
24	Fourth panel	−0.164	+0.164	+0.394	+0.732	−1.126	−2.877	−3.303	+6.180
25	Final moments	−36.479	+36.479	+29.927	−46.263	+16.336	+28.653	−4.693	−23.960

[a] Moments resulting from the elasticity of supports. [b] Final moments resulting from translation, assuming joints D and H to be fixed against rotation. [c] Final moments resulting from applied load, assuming joints D and H fixed against rotation and all joints fixed against vertical movement.

TABLE 13.—(*Continued*)

Line	Joint H			Joint J		Joint A		Joint C	
	HF	HG	HJ	JH	JI	AB	AC	CA	CD
				(*a*) Steps (*g*) and (*i*) (*Continued*)					
1	....	....	....	....	....	....	+1.6156	+1.8602	....
2	−0.0123	−0.0184	−0.0013	−0.0027	+0.0027	−0.4237	+0.4237	+0.1469	−0.9467
3	....	....	....	....	....	−1.6156	....	....	....
4	−0.0123	−0.0184	−0.0013	−0.0027	+0.0027	−2.0393	+2.0393	+2.0071	−0.9467
5	+0.0738	+0.0040	+0.0003	+0.0006	−0.0006	+0.0154	−0.0154	+0.1140	−0.2496
6	−0.0154	−0.0156	−0.0151	−0.0046	+0.0046	−0.0001	+0.0001	+0.0002	+0.0002
7	+0.0461	−0.0300	−0.0161	−0.0067	+0.0067	−2.0240	+2.0240	+2.1213	−1.1961
				(*b*) Steps (*g*) and (*i*) (*Continued*)					
8	....	....	....	....	....	....	....	....	....
9	+0.0659	....	....	....	....	....	....	....	....
10	−0.5706	+0.0864	+0.0064	+0.0128	−0.0128	+0.2992	−0.2992	−1.0385	−0.9482
11	−0.5047	+0.0864	+0.0064	+0.0128	−0.0128	+0.2992	−0.2992	−1.0385	−0.9482
12	+0.0710	+0.0038	+0.0003	+0.0006	−0.0006	+0.0148	−0.0148	+1.1097	−0.2402
13	+0.1125	+0.1140	+0.1103	+0.0336	−0.0336	+0.0004	−0.0004	−0.0014	−0.0013
14	−0.3212	+0.2042	+0.1170	+0.0470	−0.0470	+0.3144	−0.3144	−0.9302	−1.1897
				(*c*) Steps (*d*), (*e*), (*f*), (*h*), and (*i*) (*Continued*)					
15	....	....	....	....	....	....	....	....	....
16	+0.103	+2.545	+0.189	+0.378	−0.378	+1.035	−1.035	−3.593	−3.280
17	+0.103	+2.545	+0.189	+0.378	−0.378	+1.035	−1.035	−3.593	−3.280
18	−0.056	−0.003	0	0	0	−0.012	+0.012	−0.086	+0.189
19	−0.928	−0.940	−0.910	−0.277	+0.277	−0.003	+0.003	+0.012	+0.011
20	−0.881	+1.602	−0.721	+0.101	−0.101	+1.020	−1.020	−3.667	−3.080
21	+1.017	−0.661	−0.356	−0.148	+0.148	−44.654	+44.654	+46.805	−26.391
22	−5.015	+3.189	+1.826	+0.734	−0.734	+4.911	−4.911	−14.527	−18.579
23	−20.473	+12.825	+7.648	+3.010	−3.010	+0.668	−0.668	−2.009	−2.486
24	+16.989	+28.505	−45.494	−45.515	+45.515	−0.230	+0.230	+0.707	+0.839
25	−8.363	+45.460	−37.097	−41.818	+41.818	−38.285	+38.285	+27.309	−49.697

Equations can be written: For joint D—

$$- 5.404\ X_L + 0.206\ Y_L = - 1.659 \qquad (29a)$$

and, for joint H—

$$+ 0.206\ X_L - 5.404\ Y_L = + 2.836 \qquad (29b)$$

Solving simultaneously: $X_L = 0.288$ and $Y_L = - 0.514$.

Step (*f*).—Multiply the moments in line 4, Table 11, by + 0.288 and enter the product in line 18, Table 13(*c*), under the same moment subscripts as those given under "(*a*) Unit Rotation of Joint D" in Table 11. Next multiply the moments in line 4, Table 11, by − 0.514 and enter the result in line 19, Table 13(*c*), under the same moment subscripts as those given under "(*b*) Unit Rotation of Joint H" in Table 11. Line 20, Table 13(*c*), gives the final moments if deformation due to vertical shear is prevented.

Step (*g*).—To compute the required shear deformations translate member AB a distance $\Delta = 24/E$ vertically upward with respect to CD. The induced

TABLE 13.—(*Continued*)

Line	Joint C	Joint E			Joint G			Joint I	
	CE	EC	EF	EG	GE	GH	GI	IG	IJ
					(*a*) Steps (*g*) and (*i*) (*Continued*)				
1	....	....	....	....	....	....	....	....	....
2	−1.0604	−0.3458	+0.1229	+0.2229	+0.0770	−0.0368	−0.0402	−0.0116	+0.0116
3	....	....	....	....	....	....	....	....	....
4	−1.0604	−0.3458	+0.1229	+0.2229	+0.0770	−0.0368	−0.0402	−0.0116	+0.0116
5	+0.1356	+0.0277	+0.0207	−0.0484	−0.0167	+0.0080	+0.0087	+0.0025	−0.0025
6	−0.0004	−0.0011	+0.0005	+0.0006	+0.0030	−0.0056	+0.0026	−0.0003	+0.0003
7	−0.9252	−0.3192	+0.1441	+0.1751	+0.0633	−0.0344	−0.0289	−0.0094	+0.0094
					(*b*) Steps (*g*) and (*i*) (*Continued*)				
8	+1.9359	+1.8543	....	....	....	....	....	....	....
9	....	....	−0.6592	−1.1951	....	....	....	....	....
10	+0.0508	+0.1470	−0.2940	+0.1470	−0.3621	+0.1728	+0.1893	+0.0545	−0.0545
11	+1.9867	+2.0013	−0.9532	−1.0481	−0.3621	+0.1728	+0.1893	+0.0545	−0.0545
12	+0.1305	+0.0266	+0.0200	−0.0466	−0.0161	+0.0077	+0.0084	+0.0024	−0.0024
13	+0.0027	+0.0080	−0.0034	−0.0046	−0.0223	+0.0410	−0.0187	+0.0025	−0.0025
14	+2.1199	+2.0359	−0.9366	−1.0993	−0.4005	+0.2215	+0.1790	+0.0594	−0.0594
					(*c*) Steps (*d*), (*e*), (*f*), (*h*), and (*i*) (*Continued*)				
15	+9.444	−8.660	−4.105	+11.547	−12.592	....	....	....	....
16	−2.571	−7.442	+3.078	+5.582	+1.928	+5.090	+5.574	+1.606	−1.606
17	+6.873	−16.102	−1.027	+17.129	−10.664	+5.090	+5.574	+1.606	−1.606
18	−0.103	−0.021	−0.016	+0.037	+0.013	−0.006	−0.007	−0.002	+0.002
19	−0.023	−0.066	+0.028	+0.038	+0.184	−0.338	+0.154	−0.021	+0.021
20	+6.747	−16.189	−1.015	+17.204	−10.467	+4.746	+5.721	+1.583	−1.583
21	−20.414	−7.043	+3.181	+3.862	+1.397	−0.759	−0.638	−0.208	+0.208
22	+33.106	+31.792	−14.625	−17.167	−6.254	+3.458	+2.796	+0.929	−0.929
23	+4.495	+12.339	+10.512	−22.851	−23.795	+13.354	+10.441	+3.530	−3.530
24	−1.546	−4.276	−3.521	+7.797	+22.597	+29.213	−51.810	−49.429	+49.429
25	+22.388	+16.623	−5.468	−11.155	−16.522	+50.012	−33.490	−43.595	+43.595

moments are:

$$M_{AC} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 34.612 + 24}{4 \times 34.612 \times 41.647 - 24 \times 24}\right) = 1.6156$$

$$M_{CA} = \frac{6 \times 15 \times 24}{24}\left(\frac{2 \times 41.647 + 24}{4 \times 34.612 \times 41.647 - 24 \times 24}\right) = 1.8602$$

$$M_{BD} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 24.000 + 24}{4 \times 24.000 \times 37.992 - 24 \times 24}\right) = 1.6879$$

$$M_{DB} = \frac{6 \times 12 \times 24}{24}\left(\frac{2 \times 37.992 + 24}{4 \times 24.000 \times 37.992 - 24 \times 24}\right) = 2.3440$$

These moments are entered in line 1 and distributed in lines 2 and 3, Table 13(*a*), assuming joints D and H fixed against rotation. Line 4, Table 13(*a*), gives the moments before releasing joints D and H. The releasing line moment for *D* from line 4, Table 13(*a*), is $+ 2.5341 - 0.4733 - 0.0123 = + 2.0485$;

and, similarly, for joint H the releasing moment is $- 0.0123 - 0.0184 - 0.0014 = - 0.0321$.

If X_{T1} and Y_{T1} are the correction factors for joints D and H, respectively, equations can be written: Let X_{T1} equal the rotation correction factor for joint D; and Y_{T1} equal the rotation correction factor for joint H. For joint D—

$$- 5.4040\ X_{T1} + 0.2058\ Y_{T1} = + 2.0485 \qquad (30a)$$

and, for joint H—

$$+ 0.2058\ X_{T1} - 5.4040\ Y_{T1} = - 0.0321 \qquad (30b)$$

Solving Eqs. 30, $X_{T1} = - 0.3794$ and $Y_{T1} = - 0.0085$.

Multiply the moments in line 4, Table 11, by $- 0.3794$ and enter the result in line 5, Table 13(*a*), under the same moment subscripts as those given under "(*a*) Unit Rotation of Joint D," in Table 11. Next multiply the moments in line 4, Table 11, by $- 0.0085$ and tabulate in line 6, Table 13(*a*), under the same moment subscripts as those given under "(*b*) Unit Rotation of Joint H," in Table 11. Line 7, Table 13(*a*), gives the moments due to the translation of AB, including the effect of releasing joints D and H.

Next, CD is translated a distance $\Delta = 24/E$ vertically upward relative to EF. The induced moments are:

$$M_{CE} = \frac{6 \times 15 \times 24}{24} \left(\frac{2 \times 36.797 + 24}{4 \times 36.797 \times 34.738 - 24 \times 24} \right) = 1.9359$$

$$M_{EC} = \frac{6 \times 15 \times 24}{24} \left(\frac{2 \times 34.738 + 24}{4 \times 36.797 \times 34.738 - 24 \times 24} \right) = 1.8543$$

$$M_{DF} = \frac{6 \times 12 \times 24}{24} \left(\frac{2 \times 35.051 + 24}{4 \times 35.051 \times 24.000 - 24 \times 24} \right) = 2.4294$$

$$M_{FD} = \frac{6 \times 12 \times 24}{24} \left(\frac{2 \times 24.000 + 24}{4 \times 35.051 \times 24.000 - 24 \times 24} \right) = 1.8588$$

These moments are written in line 8, and distributed in lines 9 and 10, Table 13(*b*), assuming joints D and H fixed against rotation. Line 11 gives the moments before releasing the joints D and H.

The releasing moment for joint D from line 11, Table 13(*b*), is: $- 0.0352 - 0.4741 + 2.4953 = + 1.9860$; and, similarly, for joint H: $- 0.5047 + 0.0864 + 0.0064 = - 0.4119$. Let X_{T2} and Y_{T2} be the rotation correction factors for joints D and H, respectively. Then equations can be written: For joint D—

$$- 5.4040\ X_{T2} + 0.2058\ Y_{T2} = + 1.9860 \qquad (31a)$$

and, for joint H—

$$+ 0.2058\ X_{T2} - 5.4040\ Y_{T2} = - 0.4119 \qquad (31b)$$

Solving Eqs. 31 simultaneously: $X_{T2} = - 0.3651$ and $Y_{T2} = + 0.0623$.

Multiply the moments in line 4, Table 11, by $- 0.3651$ and tabulate in line 12, Table 13(*b*), under the same moment subscripts as those given under

"(*a*) Unit Rotation of Joint D," in Table 11. Next multiply the moments in line 4, Table 11, by + 0.0623 and tabulate in line 13, Table 13(*b*), under the same moment subscripts as those given under "(*b*) Unit Rotation of Joint H," in Table 11. Line 14, Table 13(*b*), gives the moments due to the translation of CD including the effect of releasing joints D and H.

The final moments due to the translation of GH a distance of $\Delta = 24/E$ vertically downward relative to EF are given in line 1, Table 14. These values

TABLE 14.—FINAL MOMENTS AFTER THE TRANSLATION OF JOINTS GH (LINE 1) AND IJ (LINE 2)

Line	Member	Joint J		Joint H			Joint F			Joint D
		JI	JH	HJ	HG	HF	FH	FE	FD	DF
1	GH......	+0.2682	−0.2682	−0.6813	−1.1426	+1.8239	+1.7433	−0.9140	−0.8293	−0.3212
2	IJ.......	−1.8635	+1.8635	+1.8627	−1.1671	−0.6956	−0.2531	+0.1353	+0.1178	+0.0461

TABLE 14.—(*Continued*)

Line	Member	Joint D (*Continued*)		Joint B		Joint I		Joint G	
		DC	DB	BD	BA	IJ	IG	GI	GH
1	GH..........	+0.2042	+0.1170	+0.0470	−0.0470	+0.3144	−0.3144	−0.9302	−1.1897
2	IJ...........	−0.0300	−0.0161	−0.0067	+0.0067	−2.0240	+2.0240	+2.1213	−1.1961

TABLE 14.—(*Continued*)

Line	Member	Joint G	Joint E			Joint C			Joint A	
		GE	EG	EF	EC	CE	CD	CA	AC	AB
1	GH......	+2.1199	+2.0359	−0.9366	−1.0993	−0.4005	+0.2215	+0.1790	+0.0594	−0.0594
2	IJ.......	−0.9252	−0.3192	+0.1441	+0.1751	+0.0633	−0.0344	−0.0289	−0.0094	+0.0094

are obtained from line 14, Table 13(*b*), as a result of the symmetry of the frame.

The final moments due to the translation of IJ a distance $\Delta = 24/E$ relative to GH are given in line 2, Table 14. These values are obtained from Table 13(*a*) as a result of symmetry.

TABLE 15.—SUM OF END MOMENTS, STEP (*h*), EXAMPLE 6

Members	Table 13(*a*)	Table 13(*b*)	Table 14	Table 14	Table 13(*c*)	End moments should be:
	Line 7	Line 14	Line 1	Line 2	Line 20	
AC and BD.......	+7.8715	−2.1941	+0.4024	−0.0611	−4.388	+132.000
CE and DF.......	−2.1931	+7.7230	−2.6503	+0.4023	−8.139	+84.000
EG and FH.......	+0.4023	−2.6503	+7.7230	−2.1931	+5.635	−60.000
GI and HJ.......	−0.0611	+0.4024	−2.1941	+7.8715	+6.684	−156.000

Step (*h*).—Let W, X, Y, and Z represent the shear correction factors for the first, second, third, and fourth panels, respectively. Coefficients for the solution are assembled as shown in Table 15 and are used for the construction of

shear equations as follows: For the first panel—

$$7.8715\,W - 2.1941\,X + 0.4024\,Y - 0.0611\,Z - 4.388 = +132.000 \quad \text{(32a)}$$

for the second panel—

$$-2.1931\,W + 7.7230\,X - 2.6503\,Y + 0.4023\,Z - 8.139 = +84.000 \quad \text{(32b)}$$

for the third panel—

$$0.4023\,W - 2.6503\,X + 7.7230\,Y - 2.1931\,Z + 5.635 = -60.000 \quad \text{(32c)}$$

and, for the fourth panel—

$$-0.0611\,W + 0.4024\,X - 2.1941\,Y + 7.8715\,Z + 6.684 = -156.000 \quad \text{(32d)}$$

Solving Eqs. 28, simultaneously: $W = 22.064$; $X = 15.617$; $Y = -11.224$; and $Z = -24.423$.

Step (*i*).—Multiply the moments in line 7, Table 13(*a*), by $W = 22.064$ and tabulate in line 21, Table 13(*c*). Multiply the moments in line 14, Table 13(*b*), by $X = 15.617$ and tabulate in line 22, Table 13(*c*). Multiply the moments in line 1, Table 14, by $Y = -11.224$ and tabulate in line 23, Table 13(*c*). Multiply the moments in line 2, Table 14, by $Z = -24.423$ and tabulate in line 24, Table 13(*c*). The addition of lines 20 to 24, inclusive, gives the final moments including the effect of joint translation.

Example 7

A multiple story transverse bent is loaded as shown in Fig. 13. The end moments of all members are required.

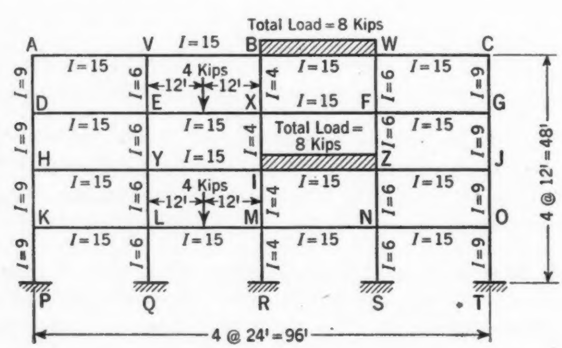


Fig. 13.—Multiple Story Transverse Bent

Step (*a*).—Joints V, W, X, Y, and Z are assumed to be secured against rotation temporarily.

Step (*b*).—Compute the stiffness factors, elastic lengths, and carry-over factors.

Step (*c*).—Rotate each of the "fixed joints" a unit amount. Then tabulate and distribute the induced moments for each "fixed joint" in a separate table.

Because of symmetry, only three tables are necessary. Compute the unbalanced moments at each of the "fixed joints" in each table.

Step (d).—Compute the end moments for the loaded members. Tabulate and distribute these moments in a fourth table.

Step (e).—Compute the releasing moments for each of the "fixed joints" from the table of step (*d*). With unbalanced moments from step (*c*), and these releasing moments, form and solve the five simultaneous equations required for the computation of the rotation correction factors for load.

Step (f).—Tabulate the rotation correction moments in the table of step (*d*).

Step (g).—Translate each story laterally an amount $\Delta = 12/E$. Tabulate and distribute the induced moments in a separate table for each story. Five simultaneous equations are required for each of the four story tables to correct for the rotation of the five "fixed joints." However, it should be noted that the coefficients of the corresponding unknowns for each set of equations are identical, the only difference being in the constants. Therefore, the four sets of equations may be solved with only slightly more labor than that involved for one set. Tabulate the rotation correction moments in the proper story tables.

Step (h).—From the final moments resulting from translation in step (*g*) shear correction equations are formed for the four stories. The solution of these four simultaneous equations gives the required shear correction factors.

Step (i).—Tabulate the shear correction moments in the table of step (*d*). The final moments are found by summing up the effect of load, rotation, and translation. Thus, the exact solution of the seventy-two unknown moments is obtained with two sets of simultaneous equations. One set consists of four equations to compensate for sidesway and the other of five equations to correct for the "fixed joints." The latter set will have a column of constants for each load considered.

Summary

A method has been developed for analyzing continuous frames with prismatic members. The extension to members having a variable cross section may be made in a manner similar to that used in previous publications.[6] The method is particularly adapted to the solution of frames such as the Vierendeel truss. The effect of flexible connections which is considered in aircraft analysis is included in the paper.

[6] "Lateral Loads and Members of Variable Section," *Bulletin No. 66*, Eng. Experiment Station, Ohio State University, Columbus, Ohio, 1931.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

FRICTION COEFFICIENTS IN A LARGE TUNNEL

BY G. H. HICKOX,[1] M. ASCE, A. J. PETERKA,[2] ASSOC. M. ASCE, AND R. A. ELDER,[3] JUN. ASCE

SYNOPSIS

Measurements made to determine friction and roughness coefficients for three different surfaces in the Apalachia Tunnel of the Tennessee Valley Authority are described in this paper. The sections tested were an 18-ft steel pipe coated with bituminous paint, an 18-ft concrete-lined tunnel, and an unlined rock tunnel of 20-ft and 22-ft nominal diameters. Discharges during the tests varied from 975 to 3,210 cu ft per sec.

Methods of reducing the data are described and the results are presented in terms of f and n in the Weisbach and Manning equations, respectively. Also, the surfaces are described in detail and illustrated so that their respective friction and roughness coefficients may be applied to other similar surfaces.

DESCRIPTION OF TUNNEL

The Apalachia Project of the Tennessee Valley Authority was built primarily as a power project and has been described by H. W. Goodhue, Assoc. M. ASCE, and R. L. Smart and A. A. Meyer,[4] Members, ASCE. The main structures are a comparatively low ponding and diversion dam, a pressure conduit approximately 8.3 miles long, a surge tank, and a powerhouse containing two turbines.

Although the pressure conduit consists mainly of a concrete-lined tunnel 18 ft in diameter, it does have two fairly long reaches of unlined rock which have nominal diameters of 20 ft and 22 ft. Just below the dam there is a comparatively short length of steel pipe 18 ft in diameter. At several other places where sufficient cover for the pressure tunnel is lacking, short lengths of steel pipe of 16-ft and 18-ft diameters are used. The alinement is nearly straight,

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **September 1, 1947**.

[1] Senior Hydr. Engr., TVA Hydr. Laboratory, Norris, Tenn.

[2] Hydr. Engr. Hydr. Laboratory, Bureau of Reclamation, Denver, Colo.

[3] Associate Hydr. Engr., TVA Hydr. Laboratory, Norris, Tenn.

[4] "The Design of Recent TVA Projects: VIII. Apalachia and Ocoee No. 3," by H. W. Goodhue, R. L. Smart, and A. A. Meyer, *Civil Engineering*, October, 1943, pp. 465–468.

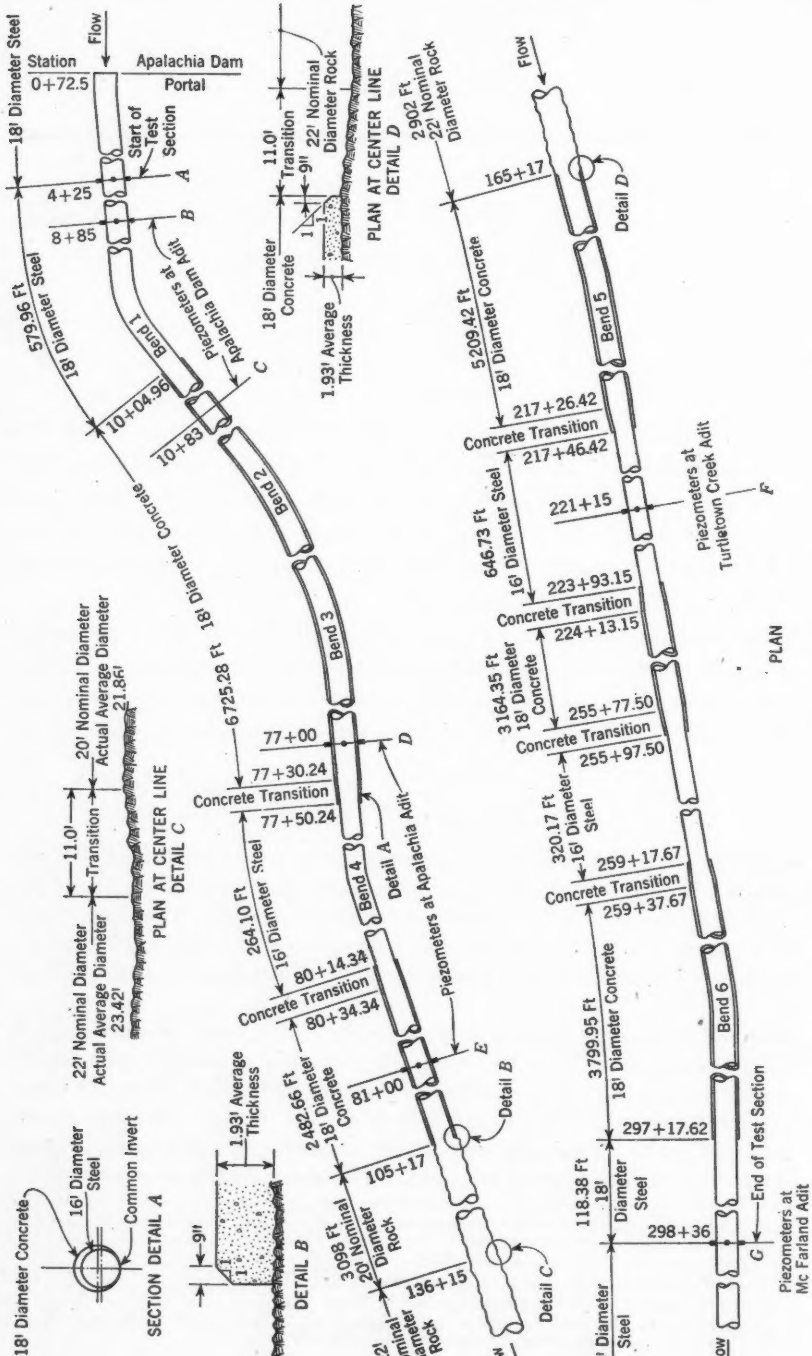

FIG. 1.—PLAN OF TEST SECTION OF APALACHIA TUNNEL SHOWING DETAILS OF LINING AND ALINEMENT

but topographical considerations necessitated several bends. In general, the grade is quite flat, sloping toward the powerhouse. Fig. 1 shows a plan of the test reach, which is approximately the upper two thirds of the conduit length. All significant features, such as lengths, diameters, types of linings, and alinement are shown.

Piezometers.—During the construction of the conduit seven sets of piezometers were installed at the locations shown in Fig. 1 so that tests could be made to determine the various hydraulic losses.

Each of these installations consisted of a ring of four piezometers spaced at 90° intervals around the conduit and manifolded together so that an average pressure could be obtained. Where the manifold ring was exposed, as on the steel pipe at the dam, the piezometers were provided with cutoff cocks so that the manifold ring could be drained to prevent damage by freezing.

Fig. 1 shows that the piezometers are so placed that friction loss measurements could be obtained on sections containing three types of conduit: 18-ft steel pipe, 18-ft concrete-lined tunnel, and unlined rock tunnel. It was also possible to measure the loss in one of the bends and in the transitions from the 18-ft tunnels to the 16-ft steel pipe.

Types of Conduit Surfaces

The types of surfaces on which friction losses were measured varied widely. Since a clear understanding of the characteristics of these surfaces is necessary for intelligent use of the data, they are described in some detail.

Steel Pipe.—The various steel pipe sections were formed of plates rolled to the proper curvature and butt welded. After the pipes were completed, the inner surfaces were covered with bituminous paint, applied hot with swabs. Each application of the swab covered an area approximately 6 in. wide and 2 ft long and deposited a layer of bituminous material approximately $\frac{1}{16}$ in. deep. The cold steel cooled the paint so rapidly that the resulting surface was quite irregular and included ridges and bumps up to $\frac{1}{8}$ in. high. However, the surface of the paint itself was almost glossy. Fig. 2 shows the interior of one of the steel pipes treated in this manner.

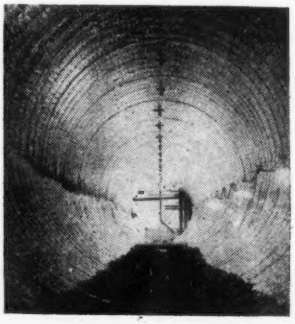

Fig. 2.—Interior of Steel Pipe Coated with Hot Bituminous Paint

Between Stations 255+97.50 and 259+17.67 the steel liner was not painted hot because of the lack of ventilation in this section at the time of painting. Instead, the bituminous paint was applied cold in a slightly diluted condition. This produced a smoother surface than that in the other pipe sections.

Concrete Tunnel.—The 18-ft concrete tunnel lining was poured in two stages. An 80° invert section was poured first. The surface of the invert, which was formed by a screed traveling along the top of the longitudinal invert forms, was floated with steel floats immediately after the screed had passed. The quality of the concrete was controlled very carefully so that the best possible surface

would result. In addition, concreting was carried on continuously so that no joint irregularities would occur in the invert section.

The arch section was poured against a three-piece steel form, of which the top piece covered a 90° sector and the two side pieces 95° each. Hinged at

FIG. 3.—CONCRETE-LINED TUNNEL SECTION AFTER REMOVAL OF STEEL FORMS

each joint, these form sections, each 5 ft long, could be collapsed and moved ahead through the other sections. Thus, irregularities due to transverse form joints occurred every 5 ft on the final surface. For convenience in erection,

FIG. 4.—COMPLETED UNLINED TUNNEL SECTION

four form sections were kept bolted together as a unit during each change. There is no reason to believe that irregularities at the unit joints are larger than any of the others, because the units were lined up by drift pins in the same manner as were the sections comprising the units. Each piece of each

section was equipped with an 18-in. by 24-in. inspection door and two or three 2-in. grout pipe holes. Each of these left a slight irregularity on the surface, averaging probably not more than $\frac{1}{32}$ in. high. Some 400 ft of forms were used so that concrete could be placed continuously without resorting to transverse bulkheads.

After the forms were removed, the surface was given a coat of curing compound. This compound disintegrates with time, however, and was probably

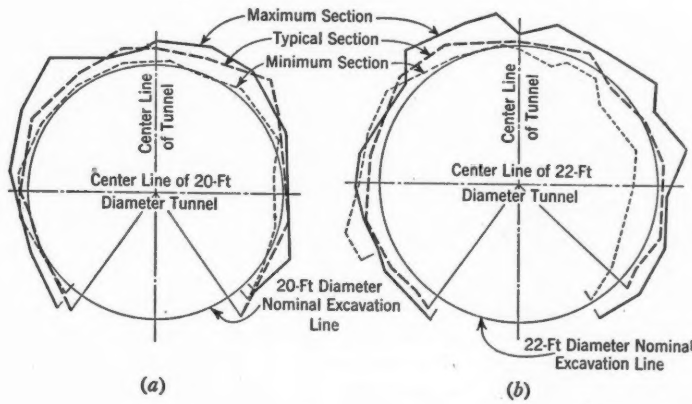

FIG. 5.—MAXIMUM, MINIMUM, AND TYPICAL SECTIONS OF UNLINED ROCK TUNNEL

completely washed off by the time the tests were made. Fig. 3 shows the concrete tunnel lining after the forms were removed and before the curing compound was applied.

Unlined Rock Tunnel.—Unlined rock tunnel sections are of two nominal diameters, but are otherwise similar. Headings were advanced by shooting drifts 11 ft long; and, although the nominal diameter was approximated at the beginning of each drift, the actual diameter gradually increased away from the heading. In addition, the unusually hard rock broke quite irregularly, as shown in Fig. 4. Maximum, minimum, and typical sections for both the 20-ft-diameter and 22-ft-diameter unlined sections are given in Fig. 5. However, the bottom of the tunnel had not been cleaned out when the cross sections were measured so that the exact shape of the bottom 80° to 90° is unknown. Areas and perimeters have been determined by measuring the known part and assuming the remainder of the cross section to be similar. Characteristics of the unlined sections are summarized in Table 1.

TABLE 1.—CHARACTERISTICS OF UNLINED ROCK TUNNEL SECTIONS

Nominal diameter (ft)	Area, in Square Feet			Diameter, equivalent circle (ft)	Hydraulic Radius, in Feet		Average overbreak (ft)
	Maximum	Minimum	Average		Equivalent	Measured	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
20...	419	335	375.3	21.86	5.465	5.36	0.93
22...	504	368	430.8	23.42	5.855	5.68	0.71

Field Tests

The field observations were made at the same time as the turbine index tests. This procedure was possible because both types of tests required that the turbines be operated over a large range of gate openings with each gate position held constant for a sufficient length of time to allow the flow to become steady.

On January 25, 1944, unit 1 was operated alone. Satisfactory pressure measurements were obtained for each 5% gate position between the 45% and 80% positions. On January 26, both units were operated identically with satisfactory pressure measurements for each 5% gate position between the 25% and 100% positions.

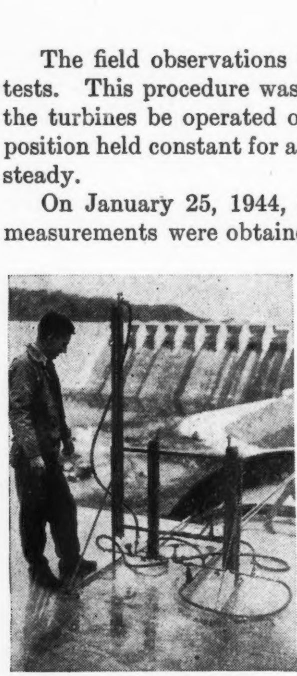

Fig. 6.—Open Mercury Manometer (Left) and Water-Air Differential Manometers (Right) at Apalachia Dam Adit

Manometer Installations.—At Apalachia Dam adit, two water-air differential manometers and one open U-tube mercury manometer were used, the last being attached to piezometer C and the others to piezometers A and B, and B and C, respectively.

At Apalachia adit, one water-air differential manometer and one open U-tube mercury manometer were used. In this location the open mercury manometer was connected to piezometer E and the water-air differential manometer to piezometers D and E.

At both Turtletown Creek adit and McFarland adit, single-column 10-ft mercury manometers were used. These manometers, constructed in the shop of the hydraulic laboratory of the Tennessee Valley Authority and carefully calibrated, were made in two sections to facilitate carrying.

At the Apalachia Dam adit, the pressures were low enough so that the gages could be installed nearly at tunnel level. Fig. 6 shows this installation. It will be noted that each manometer connection was equipped with a blowoff cock so that the piezometer lines could be flushed frequently.

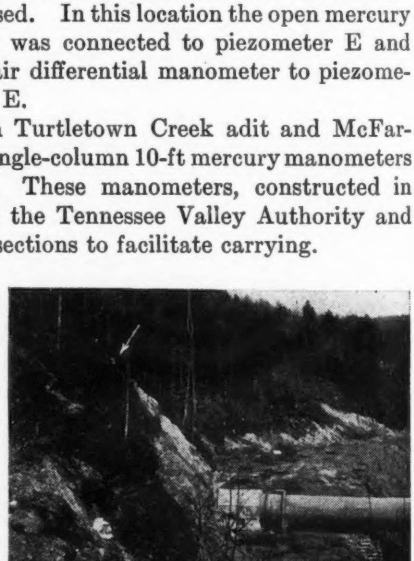

Fig. 7.—Manometer Installation (Arrow) on Hillside at Turtletown Creek Adit

Maximum and minimum pressures at the other three locations were too great to allow the manometers to be installed at tunnel level. Therefore, it was necessary to place them on the hillsides above the tunnel. Fig. 7, which shows the location of the manometer at the Turtletown Creek adit, illustrates the difficulty encountered in finding suitable manometer sites at the proper elevations. Data from all manometers were tied together through the level system established for the tunnel construction.

Connection of the manometers to the tunnel was by $\frac{3}{4}$-in. pipe. The long lines used for most of these connections could have been a source of considerable trouble due to differential temperature effects. Fortunately, however, the tests were conducted on cloudy days and air temperatures were within a few degrees of the water temperature. Thus, it was not necessary to correct for changes in density of the water in the connecting lines.

Observations.—Because each of the four locations was entirely isolated from the others and from the powerhouse, it was necessary to take readings at 1-min intervals continuously throughout the test period to insure a complete record. All observers' watches were carefully compared at the beginning and end of

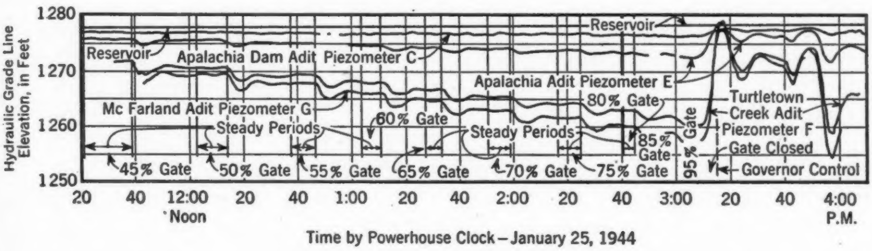

FIG. 8.—HYDRAULIC GRADE LINE ELEVATIONS FOR OPERATION OF TURBINE UNIT 1, SHOWING STEADY PERIODS FOR WHICH DATA WERE USED IN CALCULATIONS OF HEAT LOSSES

each day's observations and all time readings were corrected to correspond with a master electric clock at the powerhouse. At each location the manometers were operated and read by an experienced member of the hydraulic laboratory's engineering staff.

Manometer readings were reduced to elevations of the pressure grade line. These elevations, as well as the differential pressures given by the water-air

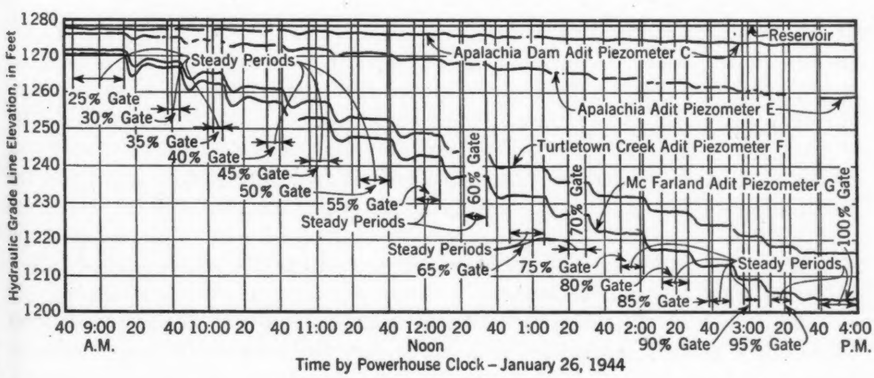

FIG. 9.

differential manometers, are shown in Table 2. Reservoir elevations for the test periods were taken from the chart of the headwater recording gage at the dam and are also shown in Table 2.

The records for both days, shown as Fig. 8 and Fig. 9, have numerous

breaks in them due to various causes. At some of the sites it was impossible to flush the lines in less than 1 min. Therefore, a break occurred every time this was done. On January 26, rain interfered several times with the recording of measurements, as the data paper became so wet that a legible record could not be kept. A long break occurred in the Turtletown record on January 26 because the top of the mercury column was behind the rubber hose used to connect the two 5-ft gage sections.

Periods during which the turbine wicket gate positions were changed were marked as were also those during which steady flow conditions existed throughout the length of the tunnel. An immediate response in pressure throughout the tunnel to a change in discharge was noted.

All observations made on each manometer within each steady flow period were averaged to give single values, which were used for all further computations. Table 2 summarizes these average values of the hydraulic grade line elevations and the differential head losses.

Tunnel Discharges

Tunnel discharges for periods of steady flow in the river were based on records obtained at a gaging station below the powerhouse. These were supplemented by interpolating discharges based on differential pressures measured on the turbine scroll cases, since steady river flow was only observed at two times during the test period due to the rapidity with which the tests were run. Steady flow conditions existed in the tunnel for an hour or more at gate openings of 25% and 100% on January 26. These periods were long enough so that flow in the river at the gaging station became stable, thus making it possible to determine the river discharges from the gage heights. Discharge measurements made at the station by the United States Geological Survey were carefully examined and replotted to give the best rating curve for the time of the friction measurements.

River stages existing at the gaging station during the 25% and 100% gate openings were found to indicate river discharges of 1,150 cu ft per sec and 3,290 cu ft per sec, respectively. These discharges included both the flow through the tunnel and the flow in the river above the powerhouse. The latter discharge was measured during the tests and found to be 84 cu ft per sec; but, as the river discharge can be determined only to the nearest 10 cu ft per sec, the net discharges through the turbines may be considered as 1,070 cu ft per sec and 3,210 cu ft per sec. It was not possible to determine other discharges from the gaging station records because no other test lasted long enough to allow the river stage to become stable.

Discharges for the remaining gate openings on January 26 and for all tests on January 25 were determined from scroll case differential pressures measured during the index tests. In addition to the differential pressures, the turbine gate openings were recorded in terms of the movement of the servomotor piston that opens the gates. This movement was measured to the nearest $\frac{1}{32}$ in. and is believed reliable.

Known relationships between piston movement, gate opening, discharge, and differential pressure were used to adjust discrepancies in the observed data.

TABLE 2.—ELEVATIONS OF HYDRAULIC GRADE LINE AND MEASURED HEAD LOSSES

Gate opening (%)	GRADE LINE ELEVATION (FEET ABOVE EL. 1200) AT PIEZOMETERS:					MEASURED LOSS, IN FEET, IN SECTIONS:		
	....	C	E	F	G	A-B	B-C	D-E
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Adit[a]	Reservoir	Apalachia Dam	Apalachia	Turtle-town Creek	McFar-land	18-ft steel pipe	Bend 1	Apalachia transition
				(a) JANUARY 25				
45	77.65	76.96	75.85	....	....	0.047	0.076	0.120
50	77.75	76.98	75.59	70.75	69.50	0.064	0.096	0.148
55	77.82	76.99	75.23	69.39	67.91	0.073	0.110	0.177
60	77.84	76.80	74.77	68.07	66.17	0.089	0.128	0.211
65	77.91	76.83	74.48	66.80	64.64	0.098	0.145	0.232
70	77.93	76.79	74.07	65.44	63.04	0.110	0.166	0.260
75	78.00	76.74	73.75	64.15	61.64	0.123	0.183	0.287
80	78.01	76.61	73.34	62.92	60.13	....	....	0.312
				(b) JANUARY 26				
25	78.38	77.64	76.29	71.46	70.51	0.051	0.088	0.145
30	78.47	77.50	75.48	68.76	67.23	0.075	0.131	0.199
35	78.55	77.31	....	65.13	62.62	0.108	0.188	....
40	78.58	77.02	73.15	61.10	57.54	0.144	0.241	0.368
45	78.62	76.72	71.93	57.13	52.98	0.177	0.297	0.450
50	78.61	76.40	70.62	52.78	47.65	0.210	0.357	0.546
55	78.66	76.08	69.32	48.61	42.59	0.251	0.411	0.634
60	78.68	75.78	67.96	....	37.24	0.290	0.475	0.724
65	78.69	75.48	66.63	39.96	31.94	0.325	0.535	0.820
70	78.70	75.11	65.33	35.86	27.02	0.363	0.592	0.899
75	78.70	74.78	64.12	31.81	22.04	0.396	0.645	0.994
80	78.69	74.46	62.84	27.91	17.28	0.432	0.699	1.066
85	78.70	74.18	61.75	24.43	13.10	0.458	0.747	1.147
90	78.69	73.92	60.76	21.28	9.14	0.489	0.788	....
95	78.67	73.71	....	18.45	5.70	0.521	0.834	..
100	78.66	73.51	....	16.72	3.68	0.536	0.862	..
100	78.65	....	59.28	16.69	3.67	....	....	1.315

[a] Names of adit where piezometers were located; or of section where loss was measured.

The actual gate opening depends on the movement of the servomotor piston. In turn, the discharge varies with the gate opening, and the differential pressure with the discharge. Thus, the differential pressure should vary smoothly with the observed movement of the servomotor piston.

It is probable that the measurement of piston travel is more accurate than the measurement of differential pressure. Accordingly, the differential pressure for each gate opening was taken from smooth curves drawn through the points determined by the observed relationship between piston movement and differential pressure. Table 3 summarizes, for each gate opening on January 25 and 26, the piston movement, the observed differential pressure, and the adjusted value taken from the smoothed curve.

The known discharge of 3,210 cu ft per sec for both units at 100% gate opening on January 26 was divided between the two units in proportion to their power outputs, which index test data showed to be 39,790 kw and 40,000

TABLE 3.—GATE POSITIONS, DIFFERENTIAL PRESSURES, AND DISCHARGES

Gate opening (%)	(a) JANUARY 25				(b) JANUARY 26								
	Unit 1				Unit 1			Unit 2			Discharge, in Cubic Feet Per Second		
	Piston movement (in.)	Differential Pressure (In. of Mercury)		Discharge (cu ft per sec)	Piston movement (in.)	Differential Pressure (In. of Mercury)		Piston movement (in.)	Differential Pressure (In. of Mercury)		Unit 1	Unit 2	Total
		Observed	Adjusted			Observed	Adjusted		Observed	Adjusted			
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
25....	4.22	1.32	1.32	525	4.25	1.345	1.35	3.19	1.360	1.36	530	530	1,060
30....	4.69	1.99	1.97	640	4.75	1.926	1.90	3.72	1.964	1.99	628	640	1,270
35....	5.19	2.76	2.74	755	5.25	2.545	2.57	4.25	2.726	2.70	730	745	1,475
40....	5.69	3.56	3.56	860	5.72	3.405	3.35	4.78	3.516	3.50	833	850	1,685
45....	6.25	4.60	4.59	975	6.25	4.255	4.25	5.28	4.313	4.31	939	943	1,880
50....	6.75	5.56	5.58	1,075	6.75	5.112	5.12	5.78	5.188	5.19	1,030	1,033	2,060
55....	7.28	6.67	6.72	1,180	7.22	5.938	5.90	6.31	6.163	6.14	1,106	1,124	2,230
60....	7.75	7.77	7.73	1,265	7.72	6.819	6.75	6.81	6.978	6.98	1,182	1,198	2,380
65....	8.25	8.84	8.80	1,350	8.22	7.455	7.58	7.31	7.883	7.83	1,252	1,269	2,520
70....	8.75	9.85	9.87	1,430	8.75	8.500	8.45	7.81	8.528	8.62	1,322	1,331	2,650
75....	9.25	10.93	10.93	1,500	9.25	9.330	9.26	8.28	9.598	9.36	1,383	1,387	2,770
80....	9.75	12.00	11.98	1,575	9.75	10.020	10.03	8.81	9.971	10.15	1,440	1,443	2,880
85....	10.22	13.09	12.95	1,640	10.25	10.620	10.73	9.31	10.905	10.86	1,490	1,495	2,980
90....	10.72	13.82	13.97	1,700	10.72	11.305	11.33	9.81	11.700	11.53	1,530	1,540	3,070
95....	....	....	....	....	11.25	12.110	11.92	10.37	12.253	12.18	1,570	1,582	3,150
100....	....	....	....	....	11.73	12.195	12.36	10.81	12.425	12.61	1,600	1,610	3,210

kw for units 1 and 2, respectively. Assuming that both units were operating at the same head and efficiency, the corresponding discharges were 1,600 cu ft per sec and 1,610 cu ft per sec. Discharges for each unit at other gate openings were then calculated by assuming discharge to be proportional to the square root of the differential pressure. These values for units 1 and 2 on January 26 are also given in Table 3, which shows the calculated discharge for 25% gate opening to be 1,060 cu ft per sec. This compares favorably with the value of 1,070 cu ft per sec obtained from the rating curve. Discharges for unit 1 on January 25 have been calculated from the same relationship and are also given in Table 3.

REDUCTION OF TEST DATA

The hydraulic grade line elevations and differential pressures given in Table 2, together with the discharge rates in Table 3 and the general data shown in Fig. 1, were sufficient for computation of the various losses in the test sections of the tunnel.

Friction losses for each of the three types of surface were expressed in terms of the friction coefficient f in the Weisbach equation:

$$h_f = f \frac{L}{D} \frac{V^2}{2g} \qquad (1)$$

and the results are plotted as a function of Reynolds' number $\mathbf{R}$, in which

$$\mathbf{R} = \frac{V D}{\nu} \qquad (2)$$

In these equations, L is the length of any test reach; D is the diameter of the pipe; V is the average flow velocity; h_f is the energy loss, in feet, over the length L; and ν is the kinematic viscosity of the flowing water. Since air and water were both very close to 60° F during the tests, the value of ν has been taken throughout the calculations as 1.2×10^{-5}, which is its value at that temperature.

Values of the roughness coefficient n in the Manning equation:

$$V = \frac{1.486}{n} R^{\frac{2}{3}} S^{\frac{1}{2}} \qquad (3)$$

were also computed. In this equation, V is the average velocity; R is the hydraulic radius; and S is the slope of the energy grade line. These results are shown on plots of n versus V.

Entrance and bend losses were expressed in terms of the velocity head by the equation:

$$H = C \frac{V^2}{2\,g} \qquad (4)$$

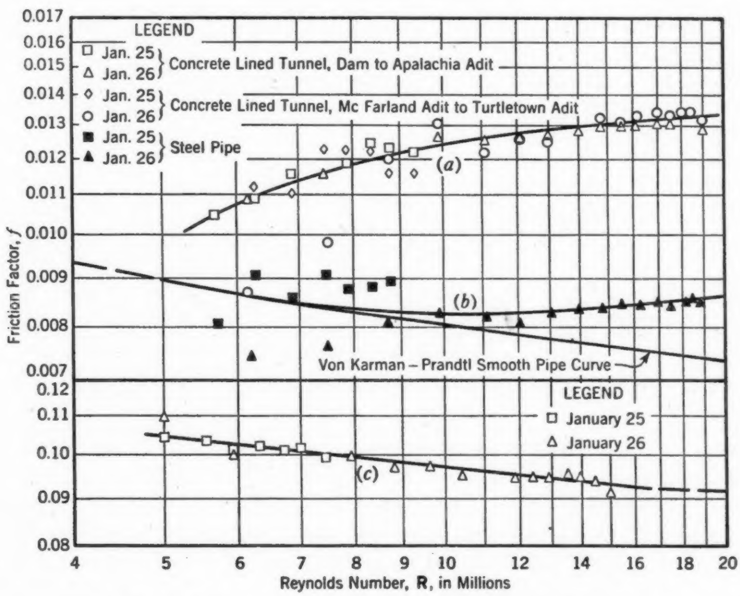

FIG. 10.—PLOT OF FRICTION FACTOR f VERSUS REYNOLDS' NUMBER R
(a) 18-Ft Bituminous-Coated Steel Pipe
(b) 18-Ft Concrete-Lined Tunnel
(c) Unlined Rock Tunnel

in which H, with an appropriate subscript, is the head loss in question; and C, with the same subscript, is the corresponding coefficient.

Losses in the tunnel transitions were expressed in terms of the difference in velocity heads by the equation:

$$H_T = C_T \frac{V^2_1 - V^2_2}{2\,g} \qquad (5)$$

In Eq. 5, V_1 and V_2 are the velocities before and after the transition, respectively.

Friction Loss in 18-Ft Steel Pipe.—Friction and roughness coefficients, f and n, for the 18-ft steel pipe were computed by Eqs. 1 and 3 from the discharges given in Table 3, the observed pressure drops given in Col. 7, Table 2, and the dimensions of the test section taken from Fig. 1. These values are given in Table 4 and are plotted as f versus **R** in Fig. 10(*a*) and as n versus V in Fig. 11(*a*).

Preliminary Bend Losses.—The pressure drop measured between piezometers B and C was composed of the loss due to bend 1 and the friction loss in 198 ft of tunnel, of which 120 ft is 18-ft steel pipe and 78 ft is 18-ft concrete-lined tunnel. As both test sections on the 18-ft concrete-lined tunnel contain bends, it was necessary to determine the bend-loss coefficients and the friction

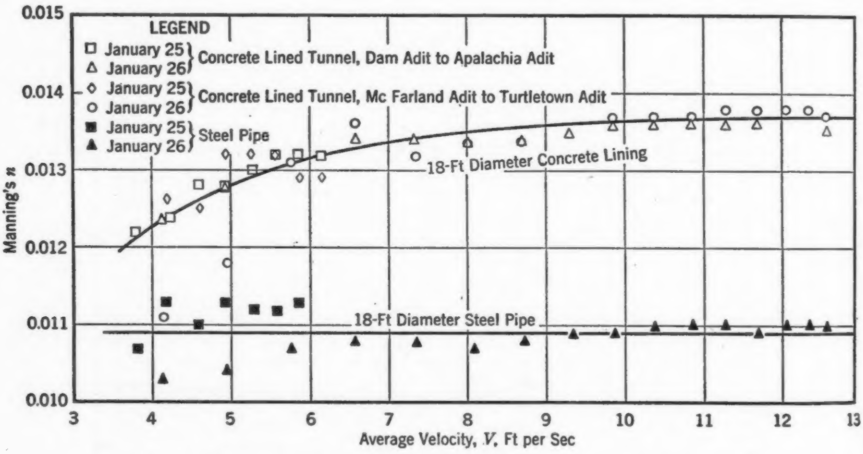

FIG. 11.—MANNING'S ROUGHNESS FACTOR n VERSUS AVERAGE VELOCITY V

coefficients for the 18-ft concrete tunnel by the method of successive approximations.

First, the tunnel between piezometers B and C was assumed to be composed entirely of 18-ft steel pipe for which the friction coefficient was known. The friction loss in this reach was then approximated by assuming it to be proportional to the measured loss between piezometers A and B. Subtracting this calculated friction loss from the total measured loss gave the loss for bend 1, from which the bend-loss coefficient C_B was computed by Eq. 4. The value of C_B thus determined was applied to bends 2 and 3 between piezometers C and D for determination of n for the concrete pipe. With this coefficient determined, the loss in bend 1 was recalculated. This change in bend loss, however, when applied to bends 2 and 3, did not measurably change f and n as first calculated for the concrete. Table 4 gives the calculated preliminary value of C_B for each rate of discharge; the average preliminary value of C_B for bend 1 is 0.253.

Table 5 lists the six bends included in the test section, together with the deflection angle, the radius, and the diameter of pipe of each. Available information concerning bend losses indicates that C_B varies with the deflection angle and the ratio of bend radius to pipe diameter. Relative values of the

TABLE 4.—Friction-Loss Computations

Gate opening (%)	Velocity (ft per sec)	Eighteen-Foot Steel Pipe			Bend Loss[a]		Eighteen-Foot Concrete-Lined Tunnel[b]					
		Manning's n	Weisbach's f	Reynolds' R	Net loss (ft)	Coefficient, C_B	Computed Loss (Ft): Bend	Computed Loss (Ft): Friction	Manning's n	Weisbach's f	Reynolds' R
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
(a) January 25											
45	3.83	0.0107	0.00805	5.7×10^6	0.056	0.246	0.11	0.88	0.0122	0.0105	5.7×10^6
50	4.22	0.0113	0.00903	6.3×10^6	0.068	0.246	0.13	1.11	0.0124	0.0109	6.3×10^6
55	4.63	0.0110	0.00857	6.9×10^6	0.079	0.237	0.16	1.42	0.0128	0.0116	6.9×10^6
60	4.97	0.0113	0.00906	7.5×10^6	0.090	0.234	0.18	1.64	0.0128	0.0116	7.5×10^6
65	5.30	0.0112	0.00877	7.9×10^6	0.103	0.236	0.21	1.91	0.0130	0.0119	7.9×10^6
70	5.61	0.0112	0.00880	8.4×10^6	0.119	0.244	0.23	2.23	0.0132	0.0124	8.4×10^6
75	5.89	0.0113	0.00892	8.8×10^6	0.130	0.241	0.26	2.44	0.0132	0.0123	8.8×10^6
80	6.19	....	....	9.3×10^6	....	....	0.28	2.68	0.0132	0.0122	9.3×10^6
(b) January 26											
25	4.16	0.0103	0.00742	6.2×10^6	0.066	0.246	0.13	1.08	0.0124	0.0109	6.2×10^6
30	4.99	0.0104	0.00758	7.5×10^6	0.099	0.256	0.18	1.64	0.0128	0.0115	7.5×10^6
35	5.79	0.0107	0.00810	8.7×10^6	0.142	0.272	0.25	....	....	....	8.7×10^6
40	6.61	0.0108	0.00829	9.9×10^6	0.179	0.264	0.32	3.18	0.0134	0.0127	9.9×10^6
45	7.38	0.0108	0.00817	11.1×10^6	0.221	0.261	0.40	3.94	0.0134	0.0126	11.1×10^6
50	8.09	0.0107	0.00806	12.1×10^6	0.267	0.262	0.49	4.74	0.0134	0.0126	12.1×10^6
55	8.75	0.0108	0.00825	13.1×10^6	0.303	0.254	0.57	5.56	0.0134	0.0127	13.1×10^6
60	9.35	0.0109	0.00835	14.0×10^6	0.350	0.258	0.65	6.45	0.0135	0.0129	14.0×10^6
65	9.89	0.0109	0.00836	14.8×10^6	0.395	0.260	0.73	7.30	0.0136	0.0130	14.8×10^6
70	10.40	0.0110	0.00844	15.6×10^6	0.436	0.259	0.80	8.08	0.0136	0.0130	15.6×10^6
75	10.88	0.0110	0.00842	16.3×10^6	0.475	0.258	0.88	8.79	0.0136	0.0130	16.3×10^6
80	11.31	0.0110	0.00850	17.0×10^6	0.513	0.258	0.95	9.60	0.0136	0.0131	17.0×10^6
85	11.70	0.0109	0.00841	17.5×10^6	0.550	0.258	1.02	10.26	0.0136	0.0131	17.5×10^6
90	12.07	0.0110	0.00845	18.1×10^6	0.578	0.255	1.08	....	....	....	18.1×10^6
95	12.38	0.0110	0.00855	18.5×10^6	0.610	0.256	1.14	....	....	....	18.5×10^6
100	12.61	0.0110	0.00848	18.9×10^6	0.631	0.256	1.18	11.73	0.0135	0.0129	18.9×10^6
Average	....	0.0109	....	....	....	0.253	....	....	....	....	....

[a] Preliminary computations of bend loss for bend 1. [b] From the dam adit to Apalachia adit.

coefficient in terms of these variables are given by William P. Creager and Joel D. Justin,[5] Members, ASCE, whose values of K, reflecting the effect of deflection angle, are given in Col. 5. Col. 6 gives values of $\frac{r}{D}$ for each bend and Col. 7 gives the corresponding values of the bend-loss coefficient for a 90° bend. The preliminary values of C_B for bends 1, 2, and 3 are given in Col. 8. For bend 1, this value is found from the calculations of Table 4. For bends 2

[5] "Hydro-electric Handbook," by William P. Creager and Joel D. Justin, John Wiley & Sons, Inc., New York, N. Y., 1927, pp. 119–120.

and 3, the value of C_B is the preliminary value for bend 1 multiplied by the ratios of appropriate factors in Cols. 5 and 7. For example, $C_{B2} = C_{B1} \times \frac{K_2}{K_1} \times \frac{F_2}{F_1} = 0.253 \times \frac{0.311}{0.571} \times \frac{0.45}{0.28} = 0.222.$

Friction Loss in 18-Ft Concrete-Lined Tunnel, Apalachia Dam Adit to Apalachia Adit.—Between Apalachia Dam adit and Apalachia adit, the tunnel is concrete lined, 18 ft in diameter, and contains two bends. Elevations of the hydraulic grade line at piezometers C and E and the differential pressures between piezometers D and E are given in Table 2. Addition of the differentials between piezometers D and E to the hydraulic grade line elevations for E gave the grade line elevations at D; subtraction of these elevations from those at C gave the friction losses between piezometers C and D.

Losses for bends 2 and 3 were calculated by Eq. 4 from the preliminary values of C_B given in Table 5, and values of n and f were calculated from Eqs. 1

TABLE 5.—BEND-LOSS COEFFICIENTS

Bend number	Deflection angle	Radius (ft)	Pipe diameter (ft)	Deflection factor, K	Ratio, $\frac{r}{D}$	Loss coefficient,[a] F	COEFFICIENT CB		Reported[b]
							Preliminary	Corrected	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1......	34° 15′ 34″	90	18	0.571	5.00	0.28	0.253	0.236	0.160
2......	15° 43′ 05″	250	18	0.311	13.9	0.45	0.222	0.207	0.140
3......	18° 18′ 53″	250	18	0.359	13.9	0.45	0.256	0.239	0.162
4......	12° 40′ 35″	90	16	0.257	5.63	0.30	....	0.114	0.077
5......	4° 41′ 51″	250	18	0.100	13.9	0.45	....	0.066	0.045
6......	8° 43′ 25″	250	18	0.179	13.9	0.45	....	0.119	0.081

[a] For a 90° bend. [b] As reported by William P. Creager and Joel D. Justin,[5] Members, ASCE.

and 3. The results of these calculations are summarized in Table 4. Fig. 10(*b*) shows f plotted against **R** and Fig. 11 gives values of n in terms of V.

Corrected Bend Losses.—Having determined the roughness coefficients for the concrete-lined tunnel, it was possible to recalculate the loss coefficients for bend 1. This calculation was made by using the average value of n determined for the steel pipe and by taking values of n for the concrete section of the tunnel from a curve drawn through the points determined for that section of the tunnel between Apalachia Dam adit and Apalachia adit. This curve was omitted from Fig. 11 as it differed so slightly from the mean curve shown there that recalculation of the bend losses on the basis of the final curve was unnecessary. From the results of these computations, summarized in Table 6, the average value of C_B for bend 1 was found to be 0.236. Corrected values of C_B for all bends are given in Col. 9, Table 5. These values are within the range commonly used for bend-loss coefficients and they lie between the values given by Messrs. Creager and Justin[5] for "safe" and "probable" values. Col. 10, Table 5, is based on the curve of probable values presented by Messrs. Creager and Justin and is the product of Cols. 5 and 7. These values are less than those based on the measurements at bend 1.

Transition Loss.—The differential manometer between piezometers D and

E at Apalachia adit measured the total head loss due to friction in 96 ft of 18-ft concrete-lined tunnel, 264 ft of steel pipe 16 ft in diameter, and 40 ft of concrete-lined transition sections having an average diameter of 17 ft, and also the loss due to bend 4 and the loss caused by the expansion from the 16-ft diameter to the 18-ft diameter.

TABLE 6.—COMPUTATION OF LOSS COEFFICIENTS

Gate opening (%)	CORRECTED BEND LOSS CALCULATIONS[a]				TRANSITION LOSS CALCULATIONS					
	Friction Loss (Ft)		Net bend loss (ft)	Bend coefficient, C_B	Computed Friction Loss (Ft)			Computed bend loss (ft)	Transition Loss	
	Steel pipe	Concrete tunnel			18-ft concrete tunnel	17-ft concrete tunnel	16-ft steel pipe		Net (ft)	Coefficient, C_T
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
(a) JANUARY 25										
45	0.013	0.010	0.053	0.232	0.013	0.008	0.053	0.042	0.004	0.029
50	0.015	0.013	0.068	0.246	0.016	0.010	0.064	0.051	0.007	0.042
55	0.019	0.016	0.075	0.225	0.020	0.012	0.077	0.061	0.007	0.035
60	0.021	0.020	0.087	0.226	0.024	0.014	0.089	0.070	0.014	0.060
65	0.024	0.023	0.098	0.224	0.028	0.016	0.101	0.080	0.007	0.027
70	0.027	0.026	0.113	0.231	0.032	0.018	0.113	0.090	0.007	0.024
75	0.030	0.029	0.124	0.230	0.035	0.020	0.125	0.098	0.009	0.028
80	0.033	0.032	....	....	0.039	0.023	0.138	0.109	0.003	0.008
(b) JANUARY 26										
25	0.015	0.013	0.060	0.223	0.016	0.009	0.062	0.049	0.009	0.055
30	0.022	0.020	0.089	0.230	0.024	0.014	0.090	0.071	0.000	0.000
35	0.029	0.027	0.132	0.253	0.034	0.020	0.121	0.096	....	....
40	0.038	0.037	0.166	0.244	0.045	0.026	0.157	0.124	0.016	0.039
45	0.047	0.046	0.204	0.241	0.057	0.033	0.196	0.155	0.009	0.018
50	0.057	0.057	0.243	0.239	0.070	0.039	0.235	0.186	0.016	0.026
55	0.067	0.066	0.278	0.234	0.082	0.047	0.276	0.218	0.011	0.015
60	0.076	0.076	0.323	0.238	0.094	0.054	0.314	0.249	0.013	0.016
65	0.085	0.086	0.364	0.239	0.106	0.060	0.352	0.278	0.024	0.026
70	0.094	0.095	0.403	0.240	0.117	0.066	0.389	0.308	0.019	0.019
75	0.103	0.104	0.438	0.238	0.128	0.072	0.425	0.336	0.033	0.030
80	0.111	0.113	0.475	0.239	0.139	0.078	0.459	0.364	0.026	0.022
85	0.119	0.121	0.507	0.238	0.149	0.084	0.493	0.390	0.031	0.024
90	0.127	0.128	0.533	0.235	0.158	0.089	0.522	0.413	....	....
95	0.133	0.135	0.566	0.238	0.166	0.094	0.549	0.435	....	....
100	0.138	0.140	0.584	0.236	0.173	0.097	0.570	0.451	0.024	0.016
Average	....	....	....	0.236	....	....	....	....	....	0.027

[a] For bend 1.

All these losses except that due to the transition can be calculated from data previously obtained. Values of roughness coefficients for the steel pipe and the concrete-lined tunnel were obtained in the same manner as those for the calculation of corrected bend loss. The difference between the total loss and the sum of the calculated friction and bend losses was taken as the loss due to expansion, the loss caused by the contraction being assumed negligible.

The value of the transition-loss coefficient C_T was calculated by Eq. 5.

The computations are summarized in Table 6, the average value of C_T for all tests being 0.027, which is 40% smaller than the value given by H. W. King,[6] Hon. M. ASCE.

Friction Loss in 18-Ft Concrete-Lined Tunnel, Turtletown Creek Adit to McFarland Adit.—Between Turtletown Creek adit and McFarland adit, the tunnel is largely a concrete-lined tunnel 18 ft in diameter like that between Apalachia Dam adit and Apalachia adit. It should, therefore, have the same friction and roughness coefficients. The computations are somewhat more difficult because the test section includes not only 6,944 ft of the 18-ft concrete lining but also 118 ft of steel pipe 18 ft in diameter, 598 ft of steel pipe 16 ft in diameter, 60 ft of concrete transitions having an average diameter of 17 ft, one contraction, two expansions, and bend 6. In addition, the tunnel does not have the same diameter at the two adits so that the measured drop in the hydraulic grade line is not equal to the drop in the energy grade line.

The drop in the energy grade line can be calculated by adding the velocity head at each piezometer to the elevation of the hydraulic grade line at that piezometer and taking the difference of these sums. Thus, if the energy grade line is represented by E and the hydraulic grade line by H, and the Turtletown Creek and McFarland adits by subscripts T and M, respectively, the drop in the energy grade line is $\Delta E = \left(H_T + \frac{V^2{}_T}{2g}\right) - \left(H_M + \frac{V^2{}_M}{2g}\right)$. This can be rewritten as $\Delta E = (H_T - H_M) + \left(\frac{V^2{}_T}{2g} - \frac{V^2{}_M}{2g}\right) = (H_T - H_M) + \Delta \frac{V^2}{2g}$. The friction loss in the 18-ft concrete-lined section of the tunnel is this value of ΔE less the other losses listed.

Loss in the steel pipe sections was calculated from Eq. 3 using n as 0.0109. Values of n for the concrete transitions with an average diameter of 17 ft were taken from the preliminary curve previously mentioned. Expansion losses were calculated using 0.027 as the value of C_T, and the loss coefficient for bend 6 was taken from Table 5. Contraction loss was neglected. Values of f and n for the 18-ft concrete were calculated from the net loss, using Eqs. 1 and 3. The computations are summarized in Table 7 and the results are plotted in Figs. 10(*b*) and 11.

Friction Loss in Unlined Rock Tunnel.—Between the Turtletown Creek and McFarland adits the tunnel consists of 3,098 ft of unlined rock tunnel 20 ft in nominal diameter, 2,902 ft of unlined rock tunnel 22 ft in nominal diameter, 7,626 ft of 18-ft concrete-lined tunnel, 20 ft of concrete-lined transition averaging 17 ft in diameter, and 369 ft of 16-ft steel pipe. Losses were also caused by bend 5, by the expansions from 18-ft concrete to 20-ft rock and from 20-ft rock to 22-ft rock, and by the contraction from 20-ft rock to 18-ft concrete. The contractions from 22-ft rock to 20-ft rock and from 18-ft concrete to 16-ft steel are so gradual that losses due to them were neglected.

Energy loss between piezometers E and F was not equal to the fall in the hydraulic grade line between the piezometers because of the smaller diameter, with corresponding higher velocity, at piezometer F. The energy loss was

[6] "Handbook of Hydraulics for the Solution of Hydraulic Problems," by H. W. King, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1939, p. 191.

determined, however, by adding the difference between velocity heads at piezometers E and F to the elevation of the hydraulic grade line at piezometer F and subtracting the sum from the elevation of the hydraulic grade line at piezometer E. This process is similar to that employed in finding the energy loss between piezometers F and G. The loss in the unlined rock tunnel was found by subtracting from the total energy loss not only the friction losses in

TABLE 7.—FRICTION LOSS IN AN 18-FT CONCRETE-LINED TUNNEL; TURTLETOWN CREEK ADIT TO MCFARLAND ADIT

Gate opening (%)	COMPUTED FRICTION LOSSES (FT)			Computed transition loss (ft)	Computed bend loss (ft)	Net loss, 18-ft concrete (ft)	Manning's n	Weisbach's f	Reynolds' R
	16-ft steel pipe	18-ft steel pipe	17-ft concrete tunnel						
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
				(a) JANUARY 25					
45	0.119	0.013	0.011	0.007	0.027	....	....	....	5.7×10^6
50	0.145	0.015	0.015	0.009	0.033	1.20	0.0126	0.0112	6.3×10^6
55	0.174	0.018	0.018	0.011	0.040	1.42	0.0125	0.0110	6.9×10^6
60	0.201	0.021	0.021	0.013	0.046	1.83	0.0132	0.0123	7.5×10^6
65	0.228	0.024	0.024	0.014	0.052	2.08	0.0132	0.0123	7.9×10^6
70	0.256	0.027	0.028	0.016	0.058	2.32	0.0132	0.0123	8.4×10^6
75	0.282	0.030	0.031	0.018	0.064	2.42	0.0129	0.0116	8.8×10^6
80	0.311	0.033	0.034	0.019	0.071	2.68	0.0129	0.0116	9.3×10^6
				(b) JANUARY 26					
25	0.141	0.015	0.014	0.009	0.032	0.90	0.0111	0.0087	6.2×10^6
30	0.202	0.021	0.021	0.013	0.046	1.46	0.0118	0.0098	7.5×10^6
35	0.272	0.029	0.030	0.017	0.062	2.42	0.0131	0.0120	8.7×10^6
40	0.355	0.037	0.039	0.022	0.081	3.44	0.0136	0.0131	9.9×10^6
45	0.442	0.046	0.049	0.028	0.101	3.99	0.0132	0.0122	11.1×10^6
50	0.531	0.056	0.059	0.033	0.121	4.95	0.0134	0.0126	12.1×10^6
55	0.624	0.065	0.070	0.039	0.142	5.81	0.0134	0.0126	13.1×10^6
60	0.710	0.075	0.080	0.044	0.162	....	....	....	14.0×10^6
65	0.795	0.084	0.090	0.050	0.181	7.74	0.0137	0.0132	14.8×10^6
70	0.880	0.092	0.100	0.055	0.200	8.53	0.0137	0.0131	15.6×10^6
75	0.960	0.101	0.109	0.060	0.219	9.43	0.0137	0.0133	16.3×10^6
80	1.037	0.109	0.118	0.065	0.236	10.27	0.0138	0.0134	17.0×10^6
85	1.113	0.117	0.126	0.070	0.253	10.94	0.0138	0.0133	17.5×10^6
90	1.18	0.124	0.134	0.073	0.269	11.72	0.0138	0.0134	18.1×10^6
95	1.24	0.131	0.141	0.077	0.283	12.31	0.0138	0.0134	18.5×10^6
100	1.29	0.136	0.146	0.080	0.294	12.58	0.0137	0.0132	18.9×10^6

the concrete sections 18 ft and 17 ft in diameter and in the 16-ft steel pipe, but also the losses due to bend 5 and the expansions and contractions.

Friction loss in the 18-ft concrete-lined tunnel was computed from Eq. 3 with the n-values taken from the smooth curve shown in Fig. 11. Friction loss due to the 20 ft of transition was also calculated by this method. In the case of the 16-ft steel pipe, however, the loss was computed with $n = 0.0109$, the value determined for the 18-ft steel pipe. Hydraulic loss due to bend 5 was handled in the same manner as was that for bends 2, 3, and 6, the value of C_B for use in Eq. 4 being taken from Table 5.

Loss in the transition from the 18-ft concrete-lined tunnel to the 20-ft unlined tunnel was calculated by Eq. 5, but the value of C_T to be used is open to

some question. On the basis of data supplied by Professor King,[6] C_T was estimated at 0.9. Similarly, based on Professor King's data from the same source and on the value of 0.027 for C_T for the expansion from the 16-ft pipe to the 18-ft concrete tunnel, C_T for the expansion from the 20-ft to the 22-ft rock section was estimated at 0.05.

At the entrance to the 18-ft lined tunnel, the concrete lining has a 9-in. chamfer, as shown in Fig. 1, detail D. Tests on concrete box culverts with a similar beveled entrance[7] indicate that the loss at entrance may be expressed by Eq. 5 with a coefficient of 0.15.

It was not possible to determine roughness coefficients for the 20-ft and 22-ft unlined tunnels separately. However, values of n and f were computed from Eqs. 3 and 1, assuming that the same values apply equally to both sections. Values of n are plotted in Fig. 12 against the average velocity for the two sections. In Fig. 10(*c*), f is plotted against a value of **R** based on the average diameters and areas of the two sections. Calculations of these factors are summarized in Table 8. Col. 5, Table 8, includes the bend loss, contraction loss, and expansion losses, all of which were calculated by expressing all velocities in terms of the velocity in the 18-ft tunnel and using that velocity for the calculations.

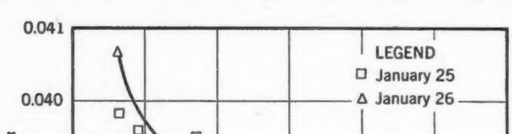

FIG. 12.—MANNING'S ROUGHNESS FACTOR n VERSUS AVERAGE VELOCITY V FOR UNLINED ROCK TUNNEL

In making the computations of f and n, the diameter was taken as that of a circle whose area was equivalent to the average area of the section. The hydraulic radius used was one fourth of the equivalent diameter. This assumption is not quite correct because of the irregularity of the tunnel cross sections. For example, the average hydraulic radii of the 20-ft and 22-ft tunnels as measured on the cross sections are actually 5.36 ft and 5.60 ft instead of 5.465 ft and 5.855 ft, respectively. Thus, use of these values in Eq. 1 produces values 2.3% smaller than those given in Table 8. Similarly, when used in Eq. 3, these values give n-values 1.5% smaller than those in Table 8. The differences are small, however; and, in view of the uncertainty of actual tunnel shapes (which depend on the character of rock and the manner in which it breaks, as well as on the care taken in measuring the irregular sections), it is believed better to use the simpler assumption of an equivalent circular area and the corresponding hydraulic radius.

SUMMARY

Tests on the Apalachia tunnel were made at discharges between 975 cu ft per sec and 3,210 cu ft per sec. Friction and roughness coefficients were de-

[7] "The Flow of Water Through Culverts," *Bulletin No. 1*, Studies in Engineering, Univ. of Iowa, Iowa City, 1926, p. 1119.

TABLE 8.—FRICTION LOSSES IN UNLINED ROCK TUNNEL

Gate opening (%)	COMPUTED FRICTION LOSSES			Computed miscel-laneous losses (ft)	Net loss, unlined tunnel (ft)	Man-ning's n	Weis-bach's f	Rey-nolds'[a] R
	18-ft concrete tunnel	17-ft concrete tunnel	16-ft steel pipe					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(a) JANUARY 25								
45	1.02	0.00	0.07	0.14	....	....	....	4.6×10^6
50	1.27	0.00	0.09	0.18	3.13	0.0398	0.104	5.0×10^6
55	1.58	0.01	0.11	0.21	3.73	0.0396	0.103	5.5×10^6
60	1.88	0.01	0.12	0.24	4.22	0.0393	0.101	5.9×10^6
65	2.17	0.01	0.14	0.28	4.82	0.0393	0.102	6.3×10^6
70	2.48	0.01	0.16	0.31	5.37	0.0393	0.101	6.7×10^6
75	2.77	0.01	0.17	0.34	5.98	0.0395	0.102	7.0×10^6
80	3.10	0.01	0.19	0.38	6.38	0.0388	0.099	7.4×10^6
(b) JANUARY 26								
25	1.24	0.00	0.09	0.17	3.17	0.0407	0.109	5.0×10^6
30	1.90	0.01	0.13	0.25	4.20	0.0390	0.100	5.9×10^6
35	2.68	0.01	0.17	0.33	....	....	....	6.9×10^6
40	3.60	0.01	0.22	0.43	7.38	0.0390	0.100	7.9×10^6
45	4.55	0.02	0.27	0.54	8.91	0.0384	0.097	8.8×10^6
50	5.53	0.02	0.33	0.65	10.69	0.0384	0.097	9.6×10^6
55	6.58	0.02	0.39	0.76	12.23	0.0380	0.095	10.4×10^6
60	7.52	0.03	0.44	0.86	....	....	....	11.1×10^6
65	8.55	0.03	0.49	0.97	15.71	0.0381	0.095	11.8×10^6
70	9.45	0.03	0.54	1.07	17.36	0.0381	0.095	12.4×10^6
75	10.34	0.04	0.59	1.17	19.06	0.0381	0.095	12.9×10^6
80	11.19	0.04	0.64	1.26	20.60	0.0381	0.096	13.5×10^6
85	11.98	0.04	0.69	1.35	21.97	0.0381	0.095	13.9×10^6
90	12.73	0.05	0.73	1.44	23.17	0.0379	0.094	14.4×10^6
95	13.40	0.05	0.77	1.52	....	....	....	14.7×10^6
100	13.90	0.05	0.80	1.57	24.78	0.0375	0.092	15.0×10^6

[a] Based on average equivalent diameter of 22.64 ft.

termined for three widely different types of surfaces, including steel coated with bituminous paint, concrete placed against steel forms, and unlined rock. Discharges were based on a rating curve established by current meter measurements supported and supplemented by observations of turbine scroll case differential pressures.

Values of f for the bituminous-coated steel pipe, at values of **R** below 8×10^6, agree with the curve suggested by Theodor von Kármán,[8,9] M. ASCE, and L. Prandtl[10] for smooth pipe. For greater values of **R**, the value of f departs from the smooth pipe curve but increases very little in actual value. The average values of f and n are about 0.0085 and 0.0109, respectively.

Values of f and n for the concrete surface placed against steel forms increase with increasing values of V and **R**, whereas values of f and n for the unlined rock tunnel decrease with increasing **R**.

[8] "Mechanical Similitude and Turbulence," by Theodor von Kármán, *Technical Memorandum No. 611*, National Advisory Committee for Aeronautics, Washington, D. C., 1931.

[9] *Proceedings*, III International Cong. on Technical Mechanics, Stockholm, 1930.

[10] "Neue Ergebnisse der Turbulenzforschung," by L. Prandtl, *Zeitschrift des Vereines Deutscher Ingenieure*, No. 5, 1933.

Bend and transition losses were approximated to enable more accurate calculation of the friction losses in the tunnel. These losses were not recalculated after the final determination of friction loss had been made; but the change, if any, would have been very small. Since these losses are a relatively small part of the total loss, the effect of a small change in the roughness coefficient used in determining them would change them only slightly and the final effect on the friction coefficients for the tunnel surfaces would be insignificant. The coefficients determined are within the range of commonly accepted values.

The data from these tests offer an opportunity to check the validity of the von Kármán equation relating friction coefficient, pipe diameter, and size of roughness. The equation, based on J. Nikuradse's[11,12] experiments on small pipes lined with sand, is

$$\frac{1}{\sqrt{f}} = 1.74 + 2 \log \frac{r}{k} \quad \text{.....................(6)}$$

in which f is the friction coefficient; r is the radius of the pipe; and k is the diameter of the sand grains composing the surface. The equation is applicable only to the region of complete turbulence in which f remains constant with increasing **R**. Although this condition was not quite reached in the tests, the constant value of f may be estimated by inspection of Fig. 10. Table 9 lists the estimated values of f and r for each type of tunnel, together with the corresponding values of $\frac{r}{k}$ and k.

TABLE 9.—COMPUTATION OF SURFACE ROUGHNESS

Tunnel surface (1)	Friction coefficient, f (2)	Pipe radius (ft) (3)	Ratio, $\frac{r}{k}$ (4)	Grain diameter, k (in.) (5)
18-ft steel......	0.0087	9.0	30,900	0.0035
18-ft concrete..	0.0135	9.0	2,690	0.04
22.6-ft rock....	0.091	11.3	6.17	22.0

Calculated values of k are in fair agreement with what would be expected after an inspection of the respective surfaces. The interior of the steel pipe was almost glassy smooth except for the irregularities caused by the method of application. The concrete lining was slightly grainy. Estimates of the mean sand grain diameter, based on visual inspection, varied from 0.03 in. to 0.06 in. The value of 0.04 is within this range. No accurate estimate of average roughness for the unlined rock can be made, but the value of 22 in. does not seem out of line after examination of Figs. 4 and 5. The agreement is the more remarkable when the difference in size between Mr. Nikuradse's apparatus and the Apalachia tunnel is considered.

[11] "Widerstandsgesetz und Geschwindigkeitsverteilung von turbulenten Wasserstromung in glatten und rauhen Rohren," *Proceedings*, III International Cong. on Technical Mechanics, Stockholm, 1930.

[12] "Stromungsgesetze in rauhen Rohren," *Forschungsheft*, Vereines Deutscher Ingenieure, Heft 361, 1933.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

STABILITY OF THIN CYLINDRICAL SHELLS IN TORSION

By R. G. Sturm,[1] M. ASCE

Synopsis

Beginning with previously derived general differential equations, this paper leads to the development of a single continuous expression for the critical shearing stress at buckling for all cylindrical shells or tubes whether long, medium, or short. This expression is reduced to a simple equation and a family of curves dependent only on the geometry of the shells and independent of the properties of the materials of which the shell is constructed. The simple equation introduces all factors dependent on the shell material. Comparisons between test results and theoretical values show excellent agreement. Methods for computing the strength of imperfect or incomplete shells are derived and compared with test results. A method of specifying maximum out-of-roundness or local discrepancies for a given percentage of strength is also proposed. A procedure for determining critical shearing stresses in thin-walled pipe lines, tanks, or fuselages is presented.

Introduction

The proper choice of sheet thickness in the design of stressed-skin construction for airplane fuselages, wings, and fuel tanks involves consideration of stability as well as of allowable stresses. Modern trends toward tubular construction for aircraft structures have emphasized the problem of selecting adequate wall thicknesses for very thin-walled tubes or partial tubes. Such members generally tend to buckle at loads far below that required to develop the ultimate strength of the material in tension, compression, or shear. Such buckling frequently results from shearing stresses induced by torque loads or transverse shear. Further interest in the stability of thin-walled tubes has resulted from the use of tubular members for actuating airplane controls. Such

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by September 1, 1947.

[1] Prof., Eng. Mechanics, and Research Prof. of Materials, Purdue Univ., Lafayette, Ind.

members may tend to "snake" and buckle laterally when subjected to torsional loads.

In the transportation of liquids by pipe lines or trucks, one of the major problems is that of selecting the proper wall thickness of the pipe or vessel containing the liquid. Although the side walls of pipe lines or tanks are by no means in simple shear, the resistance to shear buckling may be the controlling factor in the determination of adequate wall thickness. In a number of instances diagonal buckles have occurred near supports on the side of truck tanks thereby indicating the presence of buckling in shear.

In the drilling of oil wells, tubes or rods are used to rotate the drill bits. Because the relative lengths of such members are great, it has been found that a stability problem is encountered in which there is a lateral buckling of the tube or rod as a whole. This type of buckling is different from the local buckling of the walls of thin cylinders or tubes subjected to torsion. A solution to the problem of lateral buckling of a long tube or rod subject to torsion was obtained by A. G. Greenhill[2] in 1883, whereas the first solution of the buckling of the walls of long thin-walled cylinders subjected to torsion, known to the writer, was published by E. Schwerin[3] in 1924.

In 1932 Eugene E. Lundquist,[4] Assoc. M. ASCE, published the results of a great number of tests on thin-walled duralumin cylinders subjected to torsion. His work has been so carefully reported that it provides an excellent basis for comparison between theoretical and experimental results. The following year, 1933, L. H. Donnell[5] published a theoretical solution to the problem of torsion buckling of thin-walled tubes. Mr. Donnell's work is based on a number of simplifications in the general theory of buckling and leads to certain expressions for the critical shearing stress in short and moderately long thin-walled tubes and to other expressions for long tubes. An excellent evaluation of this theory has been presented by S. Timoshenko.[6] Professor Timoshenko recommends that the Donnell theory be used for short tubes but for long tubes he recommends other formulas.

Curves representing the shear buckling strength of thin-walled tubes plotted against the ratio of length to diameter show a sudden break between Mr. Donnell's theory for "short and moderately long tubes" and that of Mr. Schwerin for "long tubes" of only slightly greater length.

Mr. Donnell's experimental results indicate buckling strengths which average only about 75% of his computed values. This discrepancy was excused on the basis of initial departure from perfect circularity of tubes tested.

The confusion resulting from the use of two formulas—one for short tubes and another for long tubes—together with the difficulty encountered in justifying the discrepancies in the values obtained by existing analysis led to the study of the torsion buckling of thin-walled cylinders presented in this paper.

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[2] "On the Strength of Shafting When Exposed Both to Torsion and to End Thrust," by A. G. Greenhill, *Proceedings*, Inst. of Mech. Engrs., London, 1883, p. 182.

[3] "Torsional Stability of Thin-Walled Tubes," by E. Schwerin, *Proceedings*, 1st International Cong. for Applied Mechanics, Delft, 1924, pp. 255–265.

[4] "Strength Tests on Thin-Walled Duralumin Cylinders in Torsion," by Eugene E. Lundquist, *Technical Note No. 427*, National Advisory Committee for Aeronautics, 1932.

[5] "Stability of Thin-Walled Tubes Under Torsion," by L. H. Donnell, *Technical Report No. 479*, National Advisory Committee for Aeronautics, 1933.

[6] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1936, p. 487.

In general this study has for its object a threefold purpose: (I) To derive, on the basis of the theory of elasticity, a general expression for the maximum torque that can be applied to a thin-walled tube without the tube buckling; (II) to compare the theoretical values of critical torque thus obtained with experimental values obtained by other investigators; and (III) to suggest ways of extending these results to practical problems involving shear buckling of thin cylinders.

PART I. DERIVATION OF FORMULAS

Procedure and Assumptions

The general procedure followed in Part I is to derive expressions for the critical torsional shearing stresses in round cylindrical shells of perfectly elastic materials and to treat departures from these conditions as corollaries of the first derivation. The argument used in determining the critical shearing stresses may be stated as follows:

> Consider the cylinder deflected into some shape such that the combined differential equations of continuity and equilibrium, together with the boundary conditions, are satisfied. If the shell is in a state of neutral equilibrium, the external forces necessary to hold the shell in the deflected position will be independent of the magnitude of the deflections as long as they are so small that they do not materially change the general shape of the shell. A slight increase in external force above this value will cause collapse. Therefore, the lowest shearing stress causing neutral equilibrium is considered as the critical shearing stress for the cylinder.

The assumptions involved in the derivation of the general equations may be enumerated as follows: (*a*) The shell is a round cylinder before buckling; (*b*) the shell is of uniform thickness throughout; (*c*) the material in the shell is homogeneous, isotropic, and perfectly elastic; and (*d*) the shell wall is so thin in proportion to the diameter that the curvature will not materially disturb the linear distribution of stress through the thickness of the shell wall—that is, the ordinary flexure formulas may be applied.

System of Coordinates.—The system of cylindrical coordinates used to define and locate any particular element of shell is indicated in Fig. 1. The entire system is based on a reference cylinder the radius of which is the average mean radius of the actual shell, and the longitudinal axis of which coincides with that of the actual shell. The origin of coordinates is taken on the cylinder of reference, at the point of maximum radial displacement of the deflected shell, and midway between the ends of the cylinder.

The y-axis is parallel to the axis of the shell, and lies on the cylinder of reference, positive toward the reader. The s-axis lies on the circumference of a right section of the cylinder of reference, positive in a clockwise direction. The coordinate is $s = r\,\theta$ in which r is the radius of the cylinder of reference, and θ is the angle subtended by s in a right section of the cylinder of reference. The z-axis coincides with the radius of the cylinder of reference, the radial displacement z being measured positive outward.

Notation.—The following letter symbols, adopted for use in the paper and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Mechanics of Solid Bodies (ASA–Z10.3–1942), prepared by

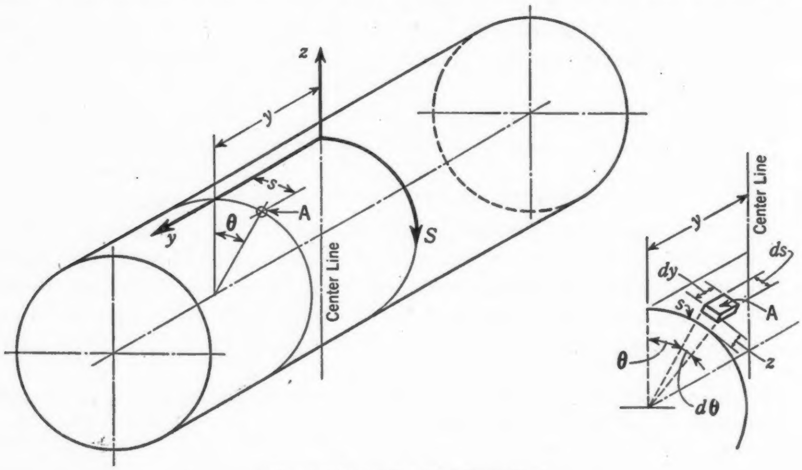

FIG. 1.—SYSTEM OF COORDINATES

a sectional committee of the American Standards Association, with Society participation, and approved by the Association in 1942:

a = an aribtrary numerical constant determined from boundary conditions and from the general differential equations; similar definitions apply to constants b, c, d, m, and n;

b = (see a);

c = the distance from the extreme fiber, in the cross section of a beam, to the neutral axis (see Eq. 28); also an arbitrary constant (see a);

D = mean diameter of a cylindrical shell, in inches;

d = (see a);

E = modulus of elasticity of the material in the shell, or in the stiffener, in pounds per square inch;

f = "a function of" (see Eq. 6a); similar definitions apply to functions g, h, j, and ϕ;

G = modulus of elasticity in shear;

g = "a function of" (Eqs. 7); also (see f) "a function of" as in Eq. 14b;

h = (see f);

I = moment of inertia per unit length of shell, in inches^3, $= t^3/12$;

j = (see f);

K = a numerical coefficient dependent on various dimensional ratios:

K_D = coefficient dependent upon L/D and D/t;

K_r = coefficient dependent upon r/L and N, introduced by the loading conditions;

L = length of the cylinder, in inches;

M = couple or moment resulting from the distribution of forces or couples on a unit length of the face of the element lying in a plane normal to a given tangent to the shell; units are in inch-pounds per inch and are positive where the force of the couple farthest away from the center of the shell is positive:

M_s = a bending couple or moment resulting from the distribution of normal forces, in a plane normal to the circumferential (s-axis) tangent;
M_y = a bending couple or moment resulting from the distribution of normal forces, in a plane normal to the longitudinal (y-axis) tangent;
M_{sy} = a twisting couple or moment resulting from the distribution of shearing forces, in a plane normal to the circumferential (s-axis) tangent;
M_{ys} = a twisting couple or moment resulting from the distribution of shearing forces, in a plane normal to the longitudinal (y-axis) tangent;

m = (see a);
N = the number of lobes into which the shell collapses;
n = (see a);
P = normal force, in pounds per linear inch, acting on a unit length of the face of an element:

P_s = force in a plane normal to the circumferential tangent to the shell;
P_y = force in a plane normal to the longitudinal tangent to the shell;

Q = a parameter; Q_1 and Q_2 are parameters determined by Eqs. 14 and 15;
Q' = a parameter; Q'_1 and Q'_2 are parameters determined by Eqs. 14 and 15;
q = any integer;
r = mean radius of a cylinder;
S = shearing force on the unit of length of the face of a given element, in pounds per linear inch:

S_{sy} = force in the y-direction, with the element in a plane normal to the circumferential (s-axis) tangent to the shell;
S_{sz} = force in the z-direction, with the element in a plane normal to the circumferential (s-axis) tangent to the shell;
S_{ys} = force in the s-direction, with the element in a plane normal to the longitudinal (y-axis) tangent to the shell;
S_{yz} = force in the z-direction, with the element in a plane normal to the longitudinal (y-axis) tangent to the shell;

s = an arc length; also axis s in Fig. 1;
t = thickness of shell wall, in inches;
u = longitudinal displacement of a point in the middle surface;
V = total vertical shear on a section, in pounds;
v = circumferential displacement of a point in the middle surface;

W = total weight, or total load uniformly distributed, in pounds (see Fig. 2);
X = a substitution factor (see Eqs. 22);
y = distances along the longitudinal (y-axis) of a tube;
z = radial deflection of a middle surface of a shell; z_o = initial departure from a round cylinder, in inches;
α = angle between the crest of a buckle wave and a longitudinal element of the cylinder (see Fig. 5);
γ = unit detrusion in a tangential plane;
Δ = displacement; a constant (indeterminate in magnitude at the time of collapse) which represents a maximum value of deflection:

Δ' = maximum initial departure from a round cylinder;
Δ_s = circumferential displacement of a point in the middle surface;
Δ_y = longitudinal displacement of a point in the middle surface;

ϵ = unit strain:

ϵ_r = unit strain in a radial direction;
ϵ_s = unit strain in a circumferential direction;
ϵ_y = unit strain in a longitudinal direction;

θ = angular distance;
μ = Poisson's ratio;
τ = tangential shear stress in the wall of the cylinder:

τ' = shear stress denoting combined bending and shear;
τ_c = critical buckling shear stress;

$$\tau_o = K_D\, E\, \frac{t^2}{D^2} \qquad (1)$$

τ_t = stress that causes local tension equal to the yield point stress;

ϕ = "a function of"; as used in Eq. 8; and
ψ = an infinitesimal angle at the center of curvature of the deflected shell, subtended by the circumferential element of length (see Eq. 4).

Deformed Element of Shell.—The element of shell considered is shown in its deformed state in Fig. 2. If u, v, and z are the displacements in the directions y, s, and z, respectively, of a point in the middle surface of the shell from the undeformed position, then the strains and the detrusion of the middle surface are defined as:

$$\epsilon_y = \frac{\partial u}{\partial y}\ ; \quad \epsilon_s = \frac{\partial v}{\partial s} + \frac{v}{r}\ ; \quad \gamma_{sy} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial y} \qquad (2)$$

By eliminating u and v, the following equation of compatibility[7,8] is obtained:

$$\frac{\partial^2 \epsilon_s}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial s^2} - \frac{\partial^2 \gamma_{sy}}{\partial s\, \partial y} - \frac{1}{r}\frac{\partial^2 z}{\partial y^2} = 0 \qquad (3)$$

[7] "Report on Arch Dam Investigation: Part VIII—Theoretical Analysis of the Structural Action of the Stevenson Creek Arch Dam," by H. M. Westergaard, *Proceedings*, ASCE, May, 1928, Pt. 3, Vol. 1, pp. 231–266.

[8] "Stress Functions for Shells," by H. M. Westergaard, *Technical Memorandum No. 351*, U. S. Bureau of Reclamation, July, 1933; "Die Stabilität der Kreiszylinderschale," *Ingenieur-Archiv.*, Vol. 3, p. 77.

It may be expressed as a function of θ, thus: $\partial\psi = \frac{\partial\psi}{\partial\theta} d\theta$, and consists of the following parts: (1) $d\theta$, original angle; (2) $-\frac{1}{r}\frac{\partial^2 z}{\partial\theta^2} d\theta$, due to change in slope over length ds; (3) $-\frac{z}{r} d\theta$, due to radial deflection; and (4) $\epsilon_s\, d\theta$, due to circumferential strain. Hence:

$$\frac{\partial\psi}{\partial\theta} = 1 - \frac{1}{r}\frac{\partial^2 z}{\partial\theta^2} - \frac{z}{r} + \epsilon_s \quad \text{.....................} (4)$$

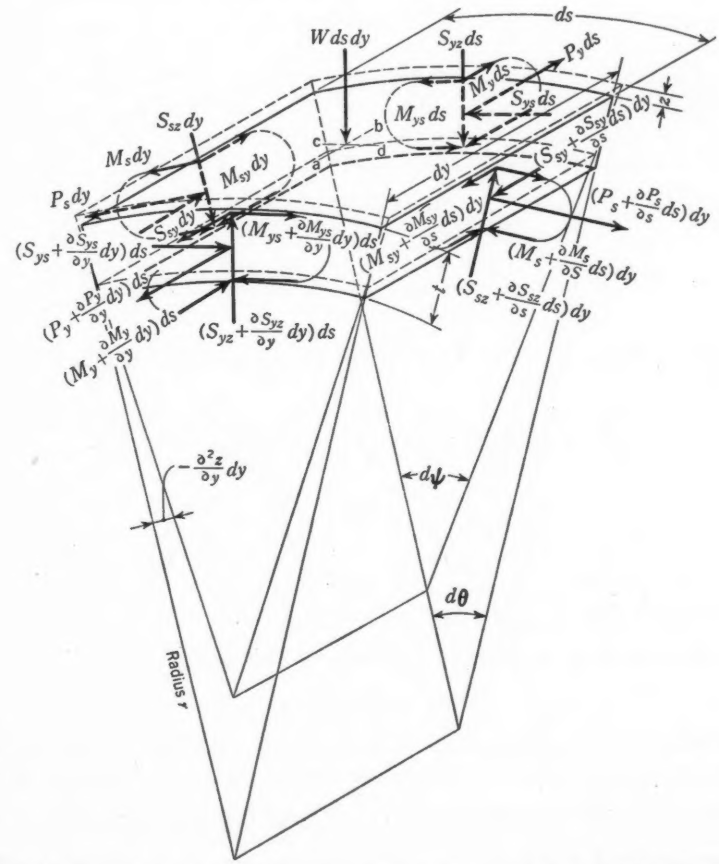

FIG. 2.—FREE-BODY DIAGRAM FOR AN ELEMENT OF A THIN SHELL (ALL FORCES AND COUPLES ARE POSITIVE IN THE DIRECTIONS SHOWN—THAT IS: z IS POSITIVE OUTWARD; y IS POSITIVE TOWARD THE READER; AND s IS POSITIVE CLOCKWISE

Eq. 4 defines an infinitesimal angle at the center of curvature of the deflected shell, subtended by the circumferential element of length.

Internal Forces and Couples.—Fig. 2 shows the forces and couples that hold an element of shell in equilibrium: An external pressure W, in pounds per

square inch; two normal forces, P_s and P_y; four shearing forces, S_{sy}, S_{sz}, S_{ys}, and S_{yz}; and four couples, M_s, M_y, M_{sy}, and M_{ys}. These concepts are defined fully in the notation.

Adaptation of General Theory of Buckling to Problem of Torsion

The derivation of a set of general differential equations governing the behavior of thin-walled cylinders subjected to any type of external load has been presented elsewhere by the writer.[9] The equations were derived from a consideration of the forces acting on a small section of a cylinder subjected to a uniform loading of any kind and of sufficient magnitude to cause the cylinder to be precisely at the point of collapse (see Fig. 1). Since all parts of the shell in Fig. 2 are in equilibrium and since the shell wall is continuous, an interrelation between the forces and deflections is obtainable. The four general differential equations derived from the conditions of equilibrium and continuity applied to the elemental part of the shell[10] are:

$$-\frac{E\,I}{1-\mu^2}\left(\frac{\partial^4 z}{\partial y^4}+\frac{\mu}{r^2}\frac{\partial^2 z}{\partial y^2}+\frac{2}{r^2}\frac{\partial^4 z}{\partial\theta^2\,\partial y^2}+\frac{1}{r^4}\frac{\partial^4 z}{\partial\theta^4}+\frac{1}{r^4}\frac{\partial^2 z}{\partial\theta^2}\right)$$
$$= W + P_s\frac{1}{r}\frac{\partial\psi}{\partial\theta} - P_y\frac{\partial^2 z}{\partial y^2} - S_{ys}\frac{1}{r}\frac{\partial^2 z}{\partial\theta\,\partial y} - S_{sy}\frac{1}{r}\frac{\partial^2 z}{\partial\theta\,\partial y}\ldots\ldots\ldots(5a)$$

$$\frac{1}{r^2}\frac{\partial^2 P_s}{\partial\theta^2}-\frac{\partial^2 P_y}{\partial y^2}-\frac{1+\epsilon_s}{r}\frac{E\,I}{1-\mu^2}\left(\frac{1}{r^4}\frac{\partial^4 z}{\partial\theta^4}+\frac{1}{r^4}\frac{\partial^2 z}{\partial\theta^2}+\frac{2-\mu}{r^2}\frac{\partial^4 z}{\partial\theta^2\,\partial y^2}\right)=0\ldots(5b)$$

$$\frac{\partial^2 P_s}{\partial y^2}+(2+\mu)\frac{\partial^2 P_y}{\partial y^2}+\frac{1}{r^2}\frac{\partial^2 P_y}{\partial\theta^2}-\frac{\mu}{r^2}\frac{\partial^2 P_s}{\partial\theta^2}=\frac{E\,t}{r}\frac{\partial^2 z}{\partial y^2}\ldots\ldots\ldots(5c)$$

and

$$\frac{\partial^2 P_s}{\partial y^2}-\frac{\mu\partial^2 P_y}{\partial y^2}=\frac{E\,t}{r}\left(\frac{\partial^3 v}{\partial y^2\,\partial\theta}+\frac{\partial^2 z}{\partial y^2}\right)\ldots\ldots\ldots\ldots(5d)$$

For the conditions existing in a tube subjected to pure torsion, certain simplifications may be made, as follows:

(1) Since there is no external pressure, $W = 0$ and

$$P_s = 0 + f(y\;s)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6a)$$

in which $f(y\;s)$ is a function of the coordinates y and s, representing the variation of the circumferential force caused by the radial deflection of the shell. When the deflection is small, $f(y\;s)$ is small, and the quantities $\frac{1}{r}\frac{\partial^2 z}{\partial\theta^2}$, ϵ_s, and $\frac{z}{r}$ which are involved in the definition of $d\psi$ are also very small compared to unity. Consequently, products of $f(y\;s)$ and these terms may be dropped from Eqs. 5a and 5b.

(2) There is no longitudinal force on the cylinder, therefore,

$$P_y = 0 + g(y\;s)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6b)$$

[9] "A Study of the Collapsing Pressure of Thin-Walled Cylinders," by R. G. Sturm, *Bulletin No. 329*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1941.

[10] *Ibid.*, Section 11, p. 17.

in which $g(y\ s)$ is a function of the coordinates y and s representing the variation in the longitudinal forces caused by the radial deflections of the shell. When the deflection is small, $g(y\ s)$ and the term $\partial^2 z/\partial y^2$ are small. Consequently, the product of $g(y\ s)$ and $\partial^2 z/\partial y^2$ may be dropped from Eq. 5*a*.

(3) Since the tube is circular, the shearing stresses will be constant except for variations caused by radial deflections of the tube and the values S_{ys} and S_{sy} will be constant and equal to $\tau\, t$ (shearing stress times thickness) except for small amounts $j(y\ s)$ and $h(s\ y)$, respectively. Consequently, the variations in S_{ys} and S_{sy} will be very small when the radial deflection, z, is small, and the term $\partial^2 z/\partial\theta\,\partial y$ is also small. The resulting products of the terms $h(s\ y)$ and $j(y\ s)$ and $\dfrac{\partial^2 z}{\partial\theta\,\partial y}$ may be dropped from Eq. 5*a*.

(4) Because of the assumed uniformity of dimensions and material in the cylinder, the average shearing stress is considered constant throughout the length of the cylinder and the displacement v is $\dfrac{S_{ys}}{t\,G}\,y$ except for small variations of $g(y\ s)$. As in previous paragraphs, products of $g(y\ s)$ and z or its derivatives may be considered negligible.

(5) Because the circumferential forces, P_s, are zero except for small variations depending on the radial deflection, the unit strain in the circumferential direction, ϵ_s, can be neglected.

Use of these five simplifications leads to the four general differential equations:

$$\frac{E\,I}{1-\mu}\left[\frac{\partial^4 z}{\partial y^4}+\frac{\mu}{r^2}\frac{\partial^2 z}{\partial y^2}+\frac{2}{r^2}\frac{\partial^4 z}{\partial\theta^2\,\partial y^2}+\frac{1}{r^4}\left(\frac{\partial^4 z}{\partial\theta^4}+\frac{\partial^2 z}{\partial\theta^2}\right)\right] + \frac{1}{r}f(s,y)=\frac{2\,S_{ys}}{r}\frac{\partial^2 z}{\partial\theta\,\partial y}\ldots\ldots\ldots\ldots(7a)$$

$$\frac{\partial^2 f}{r^2\,\partial\theta^2}-\frac{\partial^2 g}{\partial y^2}-\frac{E\,I}{r(1-\mu^2)}\left[\frac{1}{r^4}\left(\frac{\partial^4 z}{\partial\theta^4}-\frac{\partial^2 z}{\partial\theta^2}\right)+\frac{(2-\mu)\,\partial^4 z}{r^2\,\partial\theta^2\,\partial y^2}\right]=0\ldots(7b)$$

$$(2+\mu)\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{r^2\,\partial\theta^2}+\frac{\partial^2 f}{\partial y^2}-\frac{\mu}{r^2}\frac{\partial^2 f}{\partial\theta^2}=\frac{E\,t}{r}\frac{\partial^2 z}{\partial y^2}\ldots\ldots\ldots\ldots(7c)$$

and

$$\frac{\partial^2 f}{\partial y^2}-\mu\frac{\partial^2 g}{\partial y^2}=\frac{E\,t}{r}\left(\frac{\partial^3 v}{\partial y^2\,\partial\theta}+\frac{\partial^2 z}{\partial y^2}\right)\ldots\ldots\ldots\ldots(7d)$$

The problem, as defined, is to find a solution of these simultaneous differential equations which will give the correct type of radial deflection of the cylinder wall and at the same time will fulfil the boundary conditions. The value of torsional shearing forces, $S_{ys}=S_{sy}=\tau\,t$, which will make the magnitude of the deflections indeterminate is the value which will produce neutral equilibrium and hence will be the critical value. The torque producing this stress is the maximum torque that can be applied to the tube without it buckling.

The boundary conditions are simply the conditions of support at the ends of the tube which depend on the fixation of the tube. The first and principal case considered is that of "simply supported ends." In this case the ends of

the deflected tube are held so as to retain their original true circularity and, at the same time, to offer no external resistance to a change in slope of a longitudinal tangent.

The particular practical condition of the tube which this case represents is that encountered in an intermediate section of three or more sections of a tube subdivided by bulkheads relatively narrow in the direction of the tube axis.

The mathematical statement of these boundary conditions is as follows: When $y = +\frac{L}{2}$, $z = 0$; and, for all values of θ, $\frac{\partial z}{\partial \theta} = 0$.

The choice of the center of coordinates midway between the ends of the cylinder or tube and at the point of maximum displacement around the shell leads to the boundary condition that $\frac{\partial z}{\partial \theta} = 0$ when $y = 0$ and $\theta = 0$.

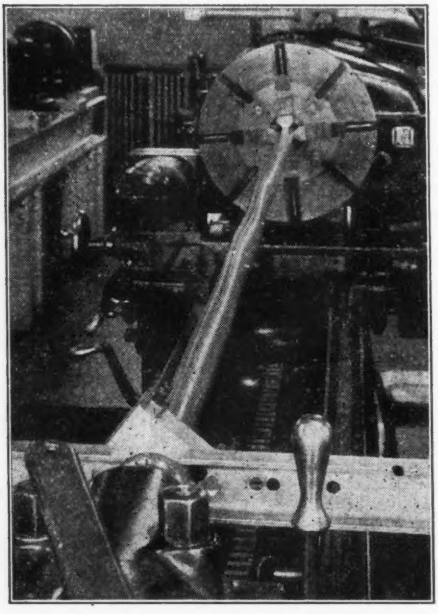

FIG. 3.—THIN-WALLED ALUMINUM ALLOY TUBE ($L/D = 44$ AND $D/t = 138$) AT NEUTRAL EQUILIBRIUM UNDER A TORSIONAL LOAD

Solution of Differential Equations.—The boundary conditions indicate that the solution of the equations is a symmetrical function with respect to both the longitudinal axis, y, and the angular axis, θ. For such high-order differential equations, solutions in the form of the product of two or more simple solutions may be sought. To obtain suggestions as to the correct type of radial deflection of the tube wall in a state of neutral equilibrium, an actual tube (Fig. 3) was twisted until it buckled and then held in its buckled state. The shape of the deflected tube clearly indicates that the general form of the deflected cross section remains essentially the same but that the points of contraflexure tend to spiral around the tube.

The general shape of the deflected tube, as shown in Fig. 3, may be represented by the equation:

$$z = \phi(y) \cos \left(N\,\theta + \frac{K_r\,\pi\,y}{L} \right) \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

In order that $\phi(y)$ may fulfil the boundary conditions (that is, zero deflection at the end of the tube) and at the same time be symmetrical about the center line of the tube:

$$\phi(y) = \Delta \cos \frac{\pi\,y}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Thus, it follows that a solution to the differential equations may be expected to be in the form of:

$$z = \Delta \cos\left(N\,\theta + \frac{K_r\,\pi\,y}{L}\right)\cos\frac{\pi\,y}{L}\ldots\ldots\ldots\ldots\ldots(10a)$$

This function (Eq. 10a) fulfils all the boundary conditions and, by proper evaluation of the constant K_r, will satisfy the differential equations. In order that the state of neutral equilibrium may be reached (that is, in order that the coefficient of the deflection term will vanish), the various derivatives—$\frac{\partial^4 z}{\partial y^4}$, $\frac{\partial^4 z}{\partial \theta^4}$, $\frac{\partial^4 z}{\partial \theta^2\,\partial y^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial \theta^2}$, and $\frac{\partial^2 z}{\partial \theta\,\partial y}$—must all have the same general functions.

Eq. 10a may be expanded for the purpose of determining the values of the derivatives; thus:

$$z = \Delta\left[\cos(N\,\theta)\cos\frac{K_r\,\pi\,y}{L}\cos\frac{\pi\,y}{L} - \sin(N\,\theta)\sin\frac{K_r\,\pi\,y}{L}\cos\frac{\pi\,y}{L}\right]\ldots(10b)$$

By substituting the following quantities—

$$m = (K_r + 1)\frac{\pi}{L}\ldots\ldots\ldots\ldots\ldots\ldots(11a)$$

and

$$n = (K_r - 1)\frac{\pi}{L}\ldots\ldots\ldots\ldots\ldots\ldots(11b)$$

—it is possible to rearrange the terms in Eq. 10a to give:

$$z = z_1 + z_2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

in which

$$z_1 = \frac{\Delta}{2}[\cos(N\,\theta)\cos(m\,y) - \sin(N\,\theta)\sin(m\,y)]\ldots\ldots(13a)$$

and

$$z_2 = \frac{\Delta}{2}[\cos(N\,\theta)\cos(n\,y) - \sin(N\,\theta)\sin(n\,y)]\ldots\ldots(13b)$$

When these values for z_1 and z_2 are substituted into Eq. 7a (which contains the value of shear, $S_{ys} = S_{sy} = \tau\,t$), it is found that they will each yield a value of τ. The function, f, may be defined in terms of K_r from Eqs. 7b, 7c, 11a, and 11b, so that the value of τ that will produce neutral equilibrium is dependent only on the constant K_r. Since there can be only one value of shearing stress which will produce neutral equilibrium, the value of τ obtained from the use of z_1 must be the same as the value of τ obtained from the use of z_2. It is possible to solve for the value of K_r that will give the same value of τ from both z_1 and z_2. This value of τ is the computed critical buckling shear stress and is designated as τ_c.

Determination of Parameters

To determine the constant parameter, K_r, it is necessary to determine the function f, but, to determine f from Eqs. 7*b* and 7*c*, it is necessary to eliminate the function, g. From Eqs. 7*b* and 7*c*, it may be noted that the functions, f and g, must have the following forms:

$$f = Q_1 z_1 + Q_2 z_2 = f_1 + f_2 \quad \text{(14a)}$$

and

$$g = Q'_1 z_1 + Q'_2 z_2 = g_1 + g_2 \quad \text{(14b)}$$

or

$$f_1 = Q_1 z_1; f_2 = Q_2 z_2 \quad \text{(15a)}$$

and

$$g_1 = Q'_1 z_1; g_2 = Q'_2 z_2 \quad \text{(15b)}$$

Since the functions, f and g, must satisfy the differential Eqs. 7*b* and 7*c*, for all values of y and θ, the parts involving z_1 and z_2 must satisfy these equations separately. The substitution of these values and the values of z_1 and z_2 from Eq. 13*b* into Eq. 7*b* gives the following relationships:

$$Q'_1 = \frac{N^2}{m^2 r^2} Q_1 + \frac{E I}{r^5 m^2 (1 - \mu^2)} [N^4 - N^2 + (2 - \mu) N^2 m^2 r^2] \quad \text{(16a)}$$

and

$$Q'_2 = \frac{N^2}{n^2 r^2} Q_2 + \frac{E I}{r^5 n^2 (1 - \mu^2)} [N^4 - N^2 + (2 - \mu) N^2 n^2 r^2] \quad \text{(16b)}$$

From Eq. 7*c* the values of Q'_1 and Q'_2 are found to be:

$$Q'_1 = \frac{Q_1 \left(\frac{\mu N^2}{m^2 r^2} - 1 \right) + \frac{E t}{r}}{2 + \mu + \frac{N^2}{m^2 r^2}} \quad \text{(17a)}$$

and

$$Q'_2 = \frac{Q_2 \left(\frac{\mu N^2}{n^2 r^2} - 1 \right) + \frac{E t}{r}}{2 + \mu + \frac{N^2}{n^2 r^2}} \quad \text{(17b)}$$

The parameter Q'_1 may be eliminated from Eqs. 16*a* and 17*a* and the value of Q_1 may be found to be:

$$Q_1 = \frac{\frac{E t}{r} - \frac{E t^3}{r^3} \frac{1}{12 (1 - \mu^2)} \left[\frac{N^4 - N^2}{m^2 r^2} + N^2 (2 - \mu) \right] \left(\frac{N^2}{m^2 r^2} + 2 + \mu \right)}{\left(\frac{N^2}{m^2 r^2} + 1 \right)^2} \quad \text{(18a)}$$

and, from Eqs. 16*b* and 17*b*, the value of Q_2 is found to be:

$$Q_2 = \frac{\frac{E t}{r} - \frac{E t^3}{r^3} \frac{1}{12 (1 - \mu^2)} \left[\frac{N^4 - N^2}{n^2 r^2} + N^2 (2 - \mu) \right] \left(\frac{N^2}{n^2 r^2} + 2 + \mu \right)}{\left(\frac{N^2}{n^2 r^2} + 1 \right)^2} \quad \text{(18b)}$$

With these equations f_1, f_2, and consequently f, are known in terms of m and n.

Equations for Critical Buckling Shear Stress

Eq. 7*a*, which contains the applied shear stress $S_{y\theta} = S_{\theta y} = \tau\, t$, may be rewritten in terms of z_1, z_2, f_1, and f_2, as follows:

$$\left\{\frac{E\,t^3}{12\,(1-\mu^2)\,r^4}\left[(m^2 r^2 + N^2)^2 - (N^2 + \mu\, m^2 r^2)\right] + \frac{Q_1}{r} - \frac{2\,N\,m}{r}\,\tau\, t\right\} z_1 + \left\{\frac{E\,t^3}{12\,(1-\mu^2)\,r^4}\left[(m^2 r^2 + N^2)^2 - (N^2 + \mu\, m^2 r^2)\right] + \frac{Q_2}{r} - \frac{2\,N\,n}{r}\,\tau\, t\right\} z_2 = 0 \quad (19)$$

In order that Eq. 19 may be true for all values of y and θ, the coefficients of z_1 and z_2 must each be zero. The resulting equations are:

$$\tau\, t = \frac{E}{12\,(1-\mu^2)}\frac{t^3}{r^3}\left[\frac{(m^2 r^2 + N^2)^2 - (N^2 + \mu\, m^2 r^2)}{2\,m\,N}\right] + \frac{Q_1}{2\,m\,N} \quad (20a)$$

and

$$\tau\, t = \frac{E}{12\,(1-\mu^2)}\frac{t^3}{r^3}\left[\frac{(n^2 r^2 + N^2)^2 - (N^2 + \mu\, n^2 r^2)}{2\,n\,N}\right] + \frac{Q_2}{2\,n\,N} \quad (20b)$$

The critical buckling shear stress τ_c is obtained when Eqs. 17 give the same value of τ. Used with Eqs. 11, they give sufficient conditions to solve for both K_r and τ_c. The values of Q_1, Q_2, m, and n may be substituted into Eq. 20*a*. The resulting equation is:

$$\frac{E\,t^3\,L}{12(1-\mu^2)\,r^3\,2\,N\,\pi}\left\{\frac{\left[(K_r+1)^2\frac{\pi^2 r^2}{L^2} + N^2\right]^2 - \left[N^2 + \mu(K_r+1)^2\frac{\pi^2 r^2}{L^2}\right]}{K_r + 1}\right\}$$
$$+ \frac{E\,t^3\,L}{12(1-\mu^2)\,r^3\,2\,N\,\pi}\left\{12(1-\mu^2)\frac{r^2}{t^2} - \left[\frac{(N^4 - N^2)L^2}{(K_r+1)^2\,\pi^2 r^2} + N^2(2-\mu)\right] \times \left[\frac{N^2}{(K_r+1)^2}\frac{L^2}{\pi^2 r^2} + 2 + \mu\right] \div (K_r+1)\left[\frac{N^2\,L^2}{(K_r+1)^2\,\pi^2 r^2} + 1\right]^2\right\} = \tau_c\, t \quad (21)$$

From Eq. 20*b*, a similar equation is obtained except that $K_r + 1$ is replaced by $K_r - 1$.

Now if the terms in the braces in Eq. 21 are designated by X, the resulting equation is:

$$\tau_c = E\,\frac{t^2}{r^2}\frac{L}{\pi\, r}\frac{X}{24(1-\mu^2)N} \quad (22a)$$

or

$$\tau_c = K_D\, E\,\frac{t^2}{D^2} \quad (22b)$$

in which

$$K_r = \frac{L}{\pi\, R}\frac{X}{6(1-\mu^2)N} \quad (23)$$

The two values of X obtained by using $K_r + 1$ and $K_r - 1$ must be the same since X is a single value. The particular values of $K_r + 1$ and $K_r - 1$ for which this condition is fulfilled may best be found by plotting values of X against values of K_r and locating the intersections of the two curves for X. In these computations, however, it is necessary to determine only one set of values of X because the values for $K_r - 1$ will be the same as the values for $K_r + 1$ except that the curve is moved two units to the right. The intersections of the pairs of curves thus obtained give values for both X and K_r.

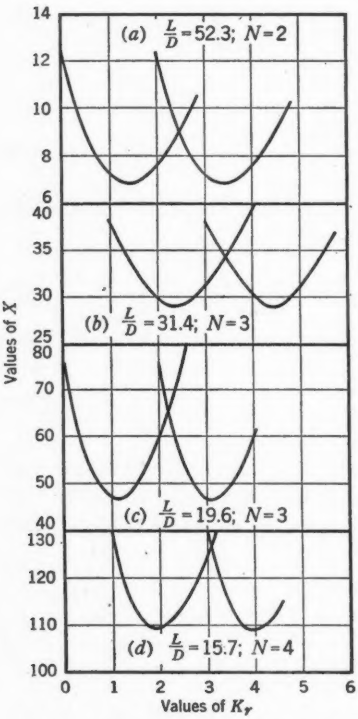

FIG. 4.—SAMPLES OF CURVES OBTAINED BY PLOTTING EQS. 18 $\left(\frac{r}{t} = 500\right)$

The values of X for any particular values of D/t and $\frac{L}{\pi\, r}$ or L/D can be computed for different integral values of N. The value of N giving the lowest resultant value of τ_e will be the number of lobes into which the tube will buckle.

In tabular computations for establishing charts it is expedient to consider given values of D/t and N and to compute values of X for a range of L/D-values in which X is a minimum or near minimum. Sample curves of X showing various types of intersections are shown in Fig. 4 and the resulting curves for K_D are shown in Fig. 5.

In subsequent considerations it will be desirable to know the angle between the crests of the buckle waves and the longitudinal elements of the cylinder. From Eq. 10*a*, it follows that, at the crest of a buckle wave, $N\,\theta + \frac{K_r\,\pi\, y}{L}$ must be a constant. Considering the buckle through the origin:

$$N\,\theta + \frac{K_r\,\pi\, y}{L} = 0 \quad\text{.....................(24a)}$$

Hence, the angle θ will change with y so that $\frac{r\,\theta}{y}$ will be the tangent of the angle, α, between the crest and a longitudinal element of the cylinder. From Eq. 24*a*, it follows that:

$$\tan\alpha = \frac{r\,\theta}{y} = -\frac{K_r\,\pi\, r}{N\,L} = -\frac{K_r\,\pi\, D}{2\,N\,L} \quad\text{.............(24b)}$$

To find the value of K_r from Fig. 5, the values of N and $\tan \alpha$ are first obtained and the value of K_r is found as:

$$K_r = \frac{2\,N}{\pi}\,\frac{L}{D}\tan \alpha \qquad (25a)$$

Eq. 22b may be rewritten as:

$$\frac{\tau_c}{E} = K_D\,\frac{t^2}{D^2} \qquad (25b)$$

All values on the right side of Eq. 25b are dependent on the geometry of the vessel and independent of the properties of the material in the shell wall

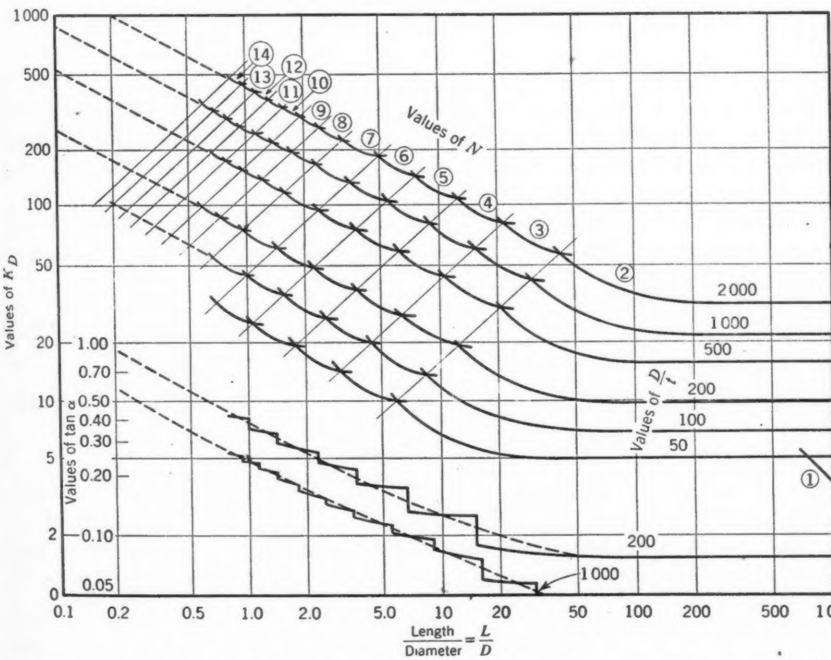

FIG. 5.—VALUES OF K_D FOR DETERMINING VALUES OF τ_c(EQ. 19) FOR THIN-WALLED CYLINDERS WITH SIMPLY SUPPORTED ENDS

except for Poisson's ratio. If E is interpreted broadly to be the effective modulus at a shearing stress τ_c, the properties of the stress-strain curve for the shell material are completely included even in the plastic range after the elastic limit of stress has been exceeded.

IMPERFECT TUBES HAVING INITIAL DEPARTURE FROM A TRUE CIRCULAR CYLINDER

If the tube considered is not a true circular cylinder but departs slightly from a true cylinder, it will begin to deflect as soon as torque is applied. In general, such initial departures from a true cylinder have no particular shape

so that, when the tube wall begins to deflect, the initial deflection will not always be additive, in its effects, to the deflection under load. However, if the initial deflection is of the same general trend as the deflection under load, the effects may be fully additive. For all practical purposes, the greatest deflections that may occur must be considered. To obtain a measure of this greatest deflection, it may be assumed that the initial departure of the tube wall from a true cylinder corresponds to the buckle pattern of the shell at neutral equilibrium.

Under this condition, the general relationship between the magnitude of the maximum deflection under load Δ, the initial deflection Δ_o, the critical shearing stress τ_c, and the actual applied stress τ may be written[9,11] as:

$$\Delta = \frac{\Delta_o \tau}{\tau_c - \tau} \qquad (26)$$

With the magnitude of the maximum deflection Δ and Eqs. 12 and 13, it is possible to determine the bending moments, and consequently the bending stresses in the shell wall, in terms of the initial deflection of the tube wall, the critical shearing stress, and the applied shearing stress.

Assuming perfect elasticity, H. M. Westergaard,[7,8,11] M. ASCE, has shown that the bending moments may be expressed in terms of the radial deflection as follows:

$$M_y = -\frac{E\,I}{1-\mu^2}\left(\frac{\partial^2 z}{\partial y^2} + \frac{\mu}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \mu\,\frac{z}{r}\right) \qquad (27a)$$

$$M_s = -\frac{E\,I}{1-\mu^2}\left(\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{z}{r^2} + \mu\,\frac{\partial^2 z}{\partial y^2}\right) \qquad (27b)$$

and

$$M_{sy} = -\frac{E\,I}{1-\mu^2}(1-\mu)\,\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta\,\partial y} \qquad (27c)$$

From each of these equations the bending moment may be computed, and from the bending moment the stress may be found by the following customary relation:

$$S = \frac{M\,c}{I} = \frac{6\,M}{t^2} \qquad (28)$$

From the use of Eqs. 27, the bending stresses in either the inside or the outside of the shell could be computed for the two coordinate directions (that is, circumferentially and longitudinally) and for a diagonal (45°) direction. With these stresses known it would be possible to construct a dyadic circle of stress[12,13] for the surface of the shell at the point considered and to determine the principal stresses. Consideration of a number of cases shows that the larger principal stress is practically in the circumferential direction and is only slightly larger

[11] "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, ASCE, Vol. LXXXV, 1922, pp. 576–654.

[12] "Advanced Mechanics of Materials," by F. B. Seely, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 37.

[13] "The Determination of Stresses from Strains on Three Intersecting Gage Lines and Its Application to Actual Tests," by W. R. Osgood and R. G. Sturm, *Journal of Research*, U. S. Bureau of Standards, Vol. 10, May, 1933.

than the maximum circumferential stress. Consequently, the circumferential stress represents a fair approximation of the controlling bending stress. Therefore, for purposes of comparing some examples of imperfect cylinders, the circumferential stress will be used as an index of the maximum bending stress in the shell wall. In cases where the critical shearing stress, τ_c, is large as compared to the elastic limit, the combined stress resulting from shear and bending should be used.

The value of z from Eqs. 12 and 13 may be substituted into Eq. 27*b* to give:

$$M_s = \frac{E\,t^3}{12(1-\mu^2)}\left\{\frac{\Delta}{2}\left(\frac{N^2-1}{r^2}+\mu\,m^2\right)[\cos(N\,\theta)\cos(m\,y)-\sin(N\,\theta)\sin(m\,y)] + \frac{\Delta}{2}\left(\frac{N^2-1}{r^2}+\mu\,n^2\right)[\cos(N\,\theta)\cos(n\,y)-\sin(N\,\theta)\sin(n\,y)]\right\}\ldots(29a)$$

The maximum value of M_s is found at $\theta = 0$, $y = 0$ and is expressed as:

$$M_s = \frac{E\,t^3}{12(1-\mu^2)}\Delta\left[\frac{N^2-1}{r^2}+\mu\,(m^2+n^2)\right]\ldots\ldots\ldots\ldots(29b)$$

The values of m and n as determined from Eqs. 11 may be substituted into Eq. 29*b* and the resulting equation may be rearranged to give:

$$M_s = \frac{\Delta E}{12(1-\mu^2)}\frac{t^3}{r^2}\left[N^2-1+2\,\mu\,\frac{\pi^2\,r^2}{L^2}(K^2_r+1)\right]\ldots\ldots(30a)$$

The value for stress may be computed from Eq. 30*a* for the value of Δ from Eq. 26 to give:

$$S = \frac{\tau_1}{\tau_c-\tau_1}\frac{E}{2(1-\mu^2)}\frac{t^2}{r^2}\frac{\Delta_o}{t}\left[N^2-1+2\,\mu\,\frac{\pi^2\,r^2}{L^2}(K^2_r+1)\right]\ldots(30b)$$

The values of N and K_r as well as the value of τ_c may be obtained from Fig. 5.

As an example to illustrate the effect of initial deflections (out-of-roundness), Eq. 30*b* is applied to a long tube of steel for which the following data apply: $L/D = 150$; $D/t = 200$; $N = 2$ (from Fig. 5); $K_r = 11.5$ (from Fig. 5); $\mu = 0.3$; $\Delta_o/t = 0.5$; $E = 30{,}000{,}000$ lb per sq in.; and the tensile yield point $= 40{,}000$ lb per sq in.

The ratio of τ_1/τ_c which will give a circumferential bending stress equal to the yield point in tension is obtained from Eq. 30*b* as $\frac{\tau_1}{\tau_c} = 0.94$. The value of $\Delta_o/t \leqq 0.5$ is easily obtained by careful fabrication except for extremely thin tubes. Therefore, the observed values of such long thin tubes should be fairly close (within 10%) to the theoretical values.

For very thin tubes of short length such as part of an aircraft fuselage a similar comparison may be made. For this case Eq. 30*b* is applied to a short thin tube of steel for which the following data are known: $L/D = 0.5$; $D/t = 1{,}000$; $N = 16$ (from Fig. 5); $K_r = 1.5$; $\mu = 0.3$; $\Delta_o/t = 0.5$; $E = 30{,}000{,}000$ lb per sq in.; and the yield point $= 40{,}000$ lb per sq in. The ratio of τ_1/τ_c

which will give a circumferential bending stress equal to the yield point in tension is $\tau_1/\tau_c = 0.81$. For exceptionally thin, short tubes, the out-of-roundness may be controlled quite easily within limits; but an out-of-roundness value of $\Delta_o/t = 0.5$ is difficult to obtain. In this case the experimental results would be expected to be less than the theoretical values by about 20%.

In view of the foregoing examples, in which a reasonable initial departure from true circularity was allowed, carefully made tests should give values of from at least 80% to 90% of the theoretical values. Actual consideration of the combined stresses extant in very thin shells indicates somewhat lower ratios for τ_1/τ_c; but actual determinations are not feasible because the properties of thin sheets of metal under biaxial stress are not known.

Eq. 30*b* permits drawing specifications for maximum allowable "out-of-roundness" (departure from true circularity) to attain at least a minimum predetermined percentage of the theoretical elastic buckling strength of the shell. The reliability of the specification, however, may be questioned until more factual data have been compiled.

Cylinders with Ends Completely Fixed

Boundary Conditions.—The boundary condition of "fixed ends" requires not only that true circularity be maintained at the ends but that the longitudinal tangent to the tube at the end be maintained parallel to the axis of the tube. The condition of absolutely fixed ends is difficult to obtain in tests and seldom, if ever, is obtained in a practical case.

The mathematical statement of these boundary conditions is as follows: When $y = \pm \frac{L}{2}$; $z = 0$; and, for all values of θ, $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial \theta} = 0$. Again the choice of the center of coordinates midway between the ends of the tube gives longitudinal symmetry. The center of coordinates is also chosen at the point of maximum displacement as before.

Type of Solution.—Again a solution consisting of the product of two functions will be sought by the same procedure as that adopted for simply supported edges. Since the characteristic form of buckling does not change, one function is chosen as before, namely: $\cos\left(N\,\theta + \frac{K_r\,\pi\,y}{L}\right)$. The other function, which previously was simply $\cos \frac{\pi\,y}{L}$, now must be a function that has both zero deflection and zero slope at the end. The following was found to be such a function: $\cos \frac{\pi\,y}{b\,L} + c \cosh \frac{\pi\,y}{L\,d}$, in which b, c, and d are arbitrary constants to be determined by the boundary conditions and one other requirement. This other requirement is that the value of τ_c is such that the coefficients of each of these terms becomes zero. This requirement leads to extremely complicated relationships.

Reference to Fig. 3 for this case reveals that for long tubes a number of waves exist so that only the end wave would be affected materially.[14] In

[14] "Strength Tests on Thin-Walled Duralumin Cylinders in Torsion," by Eugene E. Lundquist, *Technical Note No. 427*, National Advisory Committee for Aeronautics, 1932, p. 12, Fig. 4.

shorter tubes the longitudinal elements of the cylinder are affected only a short distance from the end. These facts suggest a consideration of a fixed ended tube as an equivalent shorter tube with simply supported ends.

Let the equivalent length be represented by $a\,L$. The critical shear then for this equivalent tube may be computed from the curves in Fig. 5 as soon as the value of a is known. To arrive at a value for a, consider a longitudinal section of the tube in its deflected position. The longitudinal half-wave length of the deflected wall may be found as the distance between points of zero deflection. In this case the deflection of the equivalent tube would be expressed by the equation:

$$z = \Delta \cos \left(N\,\theta + \frac{K_r\,\pi\,y}{a\,L} \right) \cos \frac{\pi\,y}{a\,L} \quad \text{(31a)}$$

The distance between points of zero deflection occurs when

$$N\,\theta + \frac{K_r\,\pi\,y}{a\,L} = q\,\pi + \frac{\pi}{2} \quad \text{(31b)}$$

in which q is any integer. Since θ does not vary, the difference between y_1 and y_2, two successive points of zero deflection, may be found from $(y_2 - y_1)\dfrac{K_r\,\pi}{a\,L} = \pi$; or

$$y_2 - y_1 = \frac{a\,L}{K_r} \quad \text{(31c)}$$

in which K_r is the value obtained for the "simply supported" tube whose length is $a\,L$. Now consider an end half wave where one point of zero deflection falls at the end of the tube. This curve will be very nearly a sine form. If the wave is replaced by another curve of the form, $1 - \cos 2\,\dfrac{\pi\,a\,L}{K_r}$, both the deflection and slope at the end would be zero. The point of contraflexure in the latter curve falls about one fourth of the way from the end to the next point of zero deflection. The tube length between these points of contraflexure is taken as the equivalent tube with simply supported ends. This length then is

$$a\,L = \left(L - \frac{2\,a\,L}{4\,K_r} \right) = \left(L - \frac{a\,L}{2\,K_r} \right) \quad \text{(32a)}$$

from which it is found that

$$a = \frac{2\,K_r}{2\,K_r + 1} \quad \text{(32b)}$$

in which K_r is based on the length $a\,L$. The value of a can be found quickly by successive approximations as follows: First, assume a value of $a\,L$; then find K_r from Fig. 5; and solve for a from Eq. 31b. With this new value of a again solve for K_r, and so on. In two or three cycles the values agree within the limits of error for reading the curves.

PART II. COMPARISON BETWEEN THEORETICAL AND EXPERIMENTAL RESULTS

Cylinders Subject to Torque

The buckling of the walls of thin-walled cylinders subjected to torque has been recognized as a problem in testing materials. In 1924, N. S. Otey[15] considered the effect of ratio of the diameter to wall thickness of specimens on the strength values obtained. In 1930 R. L. Templin, M. ASCE, and R. L. Moore, Assoc. M. ASCE, considered this phenomenon[16] from the point of view of both diameter-to-thickness ratios and length-to-diameter ratios of test specimens for determining the shearing yield strength of materials.

The first results of extensive tests to determine the buckling strength of very thin-walled cylinders were published by Mr. Lundquist[4] in 1932. The specimens for these tests were thin duralumin sheets rolled into a tube, with edges joined by bolting them to a splice plate of the same thickness as the sheets. Diameter-to-thickness ratios ranged from about 700 to about 2,800—in the range of aircraft fuselage proportions. The tubes were large in diameter (from 15 in. to 30 in.) but were relatively short (L/D-values from 0.1 to 2.5).

The following year (1933) Mr. Donnell[5] published some test results to support his theory. The specimens in these tests were also formed from thin sheets of steel and brass shim stock rolled into tubes. The edges were lapped and soldered, which gave a longitudinal seam of double thickness on one side of the tube. Mr. Donnell's specimens were mostly with small tubes (from 0.318 in. to 5.88 in. in diameter for all except one specimen). The diameter-to-thickness ratios were not as high as Mr. Lundquist's but the L/D-ratios were higher.

In 1931 Mr. Moore made some tests at the Aluminum Research Laboratories in New Kensington, Pa., on specimens machined from extruded and drawn tubing. The length-to-diameter ratios ranged from 5 to 7 and the diameter-to-thickness ratios from about 10 to 100. The results of these tests are included in a report by A. H. Stang, W. Ramberg, and G. Back.[17]

In 1939 another series of tests designed to cover the range from short to moderately long tubes was made by Mr. Moore and D. A. Paul.[18] The material used was aluminum alloy 51S–T, whose typical mechanical properties are: Yield strength, 40 kips per sq in.; tensile strength, 48 kips per sq in.; and elongation (for a thin sheet) 14% in 2 in. The tubes were extruded and drawn to size. They were very true to nominal dimensions. The maximum variation in wall thickness was less than ±5%, and the variation in diameters was less than 0.1%. The ends of the test specimens were closed with steel plugs machined to fit the inside diameter of the tubes so that, when the ends of the tubes were gripped in the jaws of the testing machine, they were quite

[15] "Torsional Strength of Nickel, Steel and Duralumin Tubing as Affected by the Ratio of Diameter to Gage Thickness," by N. S. Otey, *Technical Note No. 189*, National Advisory Committee for Aeronautics, 1924.

[16] "Specimens for Torsion Tests of Metals," by R. L. Templin and R. L. Moore, *Proceedings*, A.S.T.M., Vol. 30, Pt. II, 1930, pp. 534–543.

[17] "Torsion Tests of Tubes," by A. H. Stang, W. Ramberg, and G. Back, *Technical Report No. 601*, National Advisory Committee for Areonautics, 1937.

[18] "Torsional Stability of Aluminum Alloy Seamless Tubing," by R. L. Moore and D. A. Paul, *Technical Note No. 696*, National Advisory Committee for Aeronautics, 1939.

effectively clamped as well as held to shape. The length-to-diameter ratios of these specimens ranged from 1 to 60 and the diameter-to-thickness ratios from about 75 to 140.

Fig. 6 shows a comparison between the measured and computed values of critical shearing stresses for two sizes of tubes. The experimental values of critical shearing stress from the Moore-Paul[18] tests generally lie above the com-

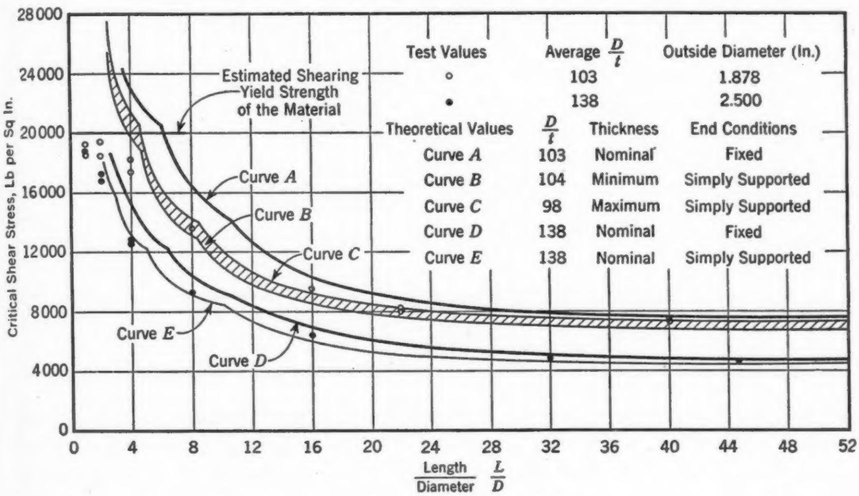

FIG 6.—COMPARISON OF THEORETICAL STRESS τ_c, WITH THE RESULTS OF THE MOORE TESTS, FOR TUBES 0.018 IN. THICK

puted curve for "simply supported ends" based on minimum measured thickness and some lie above the computed curve for "simply supported ends" based on maximum measured thickness. In all cases the measured values lie below the computed curves for fixed ends. In the range of low L/D-ratios, where the critical shearing stress in the tube approaches the shearing yield strength of the material, the measured values lie appreciably below the computed curves. This condition may be explained by a consideration of two factors, either or both of which may be responsible for this tendency.

As the material is stressed near the yield strength in shear, plastic action takes place and the shearing deformations increase rapidly. This larger deformation would have much the same effect as a higher stress on the behavior of the tube. The shearing-stress, shearing-strain curves for most materials, including aluminum alloys, are relatively very flat at stresses above the yield strength.

The second factor influencing the critical shearing stress as determined by tests is the unavoidable initial departure from true circularity. From Eq. 30*b* it is found that, for a given value of Δ_o/t and a given maximum stress S, the ratio τ_1/τ_c will be less for short tubes in which N is large and the value of τ_c is high. For long tubes, however, the ratio of τ_1/τ_c would be expected to be much closer to unity. A comparison between the computed and measured critical shearing stresses substantiates this reasoning.

Fig. 7 shows a comparison between computed values of critical shearing stress and values taken from Mr. Lundquist's work. As previously mentioned,

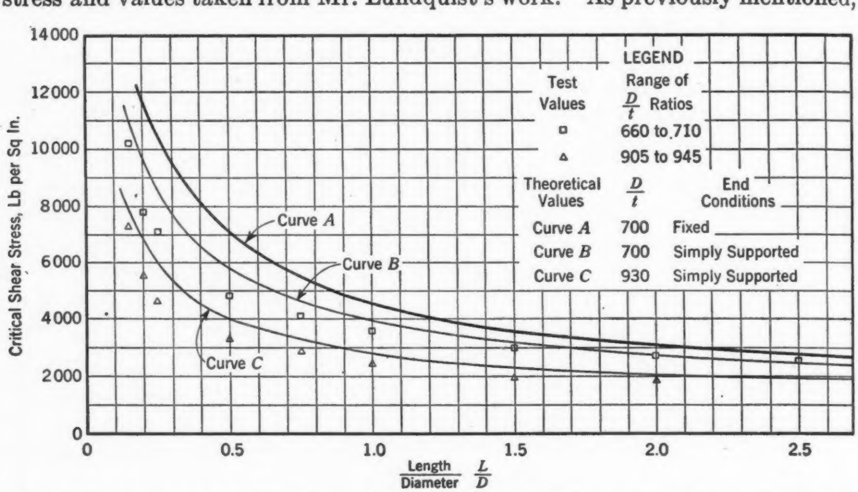

FIG. 7.—COMPARISON OF THEORETICAL STRESS τ_c WITH THE RESULTS OF THE LUNDQUIST TESTS

the specimens each contain a longitudinal seam. Such a seam may have two effects: (*a*) It may introduce an eccentricity that would tend to cause low values of measured critical shearing stress; or (*b*) it may serve as a longitudinal

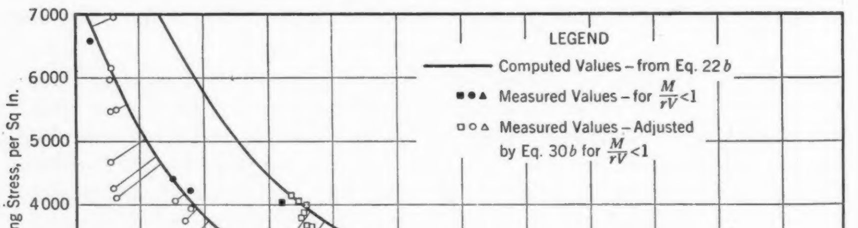

FIG. 8.—COMPARISON OF COMPUTED AND MEASURED VALUES OF CRITICAL BUCKLING SHEARING STRESS FOR TRANSVERSE SHEAR AND BENDING

stiffener and may thereby tend to strengthen the tube.

The fact that Mr. Lundquist reported that the first visible indication of buckling (relatively large radial deflection) occurred at a load well below that

causing failure suggests that the eccentricities were relatively large compared to the wall thickness. Fig. 8 reveals that the measured values of critical shearing stress, as reported by Mr. Lundquist,[19] are relatively lower with respect to the computed values than were the values determined from the seamless tubes used by Messrs. Moore and Paul.[18] In general, however, the agreement is quite satisfactory when allowances for out-of-roundness are considered.

Shear and Bending.—Results from tests of thin-walled duralumin cylinders in combined transverse shear and bending which Mr. Lundquist reported in 1935[19] serve as an excellent basis for a check on the extended applicability of this theory to shear failure of thin-walled cylindrical vessels subjected to shear loads other than torsion. Inasmuch as the longitudinal seam was always placed on the extreme tension side, it would not be expected to influence the shear distribution or the curvature at the sides of the cylinders. Fig. 8 shows a comparison of the measured and computed values of critical shearing stress using Eq. 33 in Part III.

PART III. APPLICABILITY OF THEORY TO OTHER PRACTICAL PROBLEMS

The analysis presented herein can be applied directly to long cylinders stiffened with circumferential rings by considering the length between stiffeners as the length of cylinder with simply supported ends. If heavy diaphragms are used, some restraint may be present; but, for design purposes, such restraint may be too indefinite to be taken into account.

Stiffened Cylinders Subject to Torque and Transverse Shear

For cylinders with both circumferential and longitudinal stiffeners, the shell may buckle, in some cases, in waves extending from one circumferential stiffener to the next without encountering a longitudinal stiffener; or it may buckle in waves that encounter a longitudinal stiffener before reaching the circumferential stiffener. If a complete buckle wave extending from one circumferential stiffener to the next is not interrupted by longitudinal stiffeners, the panel might reasonably be considered as buckling at the same stress as if the longitudinal stiffeners were not there. If the crest of a buckle wave intersects two longitudinal stiffeners, however, it may be assumed that the longitudinal distance between intersections is the effective length of the cylinder. The torsional buckling strength, then, of the stiffened cylinder would be that of an unstiffened cylinder of this shorter length. The angle α used for determining the reduced effective length should agree with the length finally computed. This computation may be made by successive approximations, because the series converges very rapidly.

Attention is called to the fact that, when such a double stiffened shell buckles, a series of trusses is formed with the buckled shell acting as a tension member similar to the web in a "tension field" girder. In this type of action both the longitudinal and circumferential stiffeners form the basic framework for carrying load.

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[19] "Strength Tests of Thin-Walled Duralumin Cylinders in Combined Transverse Shear and Bending," by E. E. Lundquist, *Technical Note No. 523*, National Advisory Committee for Aeronautics, April, 1935.

Although no adequate method has been developed for determining, by direct analysis, the critical maximum shear for cases of variable shear, it is possible to obtain a fair indication of this critical shear by suitable assumptions. One possible application of the results of this analysis to a variable shear problem is its use in computing the critical shear in the side walls of a thin-walled cylindrical vessel acting as a beam.

In many practical cases the bending moment in the vessel is small particularly where the shear is greatest. In such cases it has been found from experience and test[19] that the small bending moment has practically no effect on the shear resistance of the vessel walls or on the character of failure obtained.

When buckling governs, the critical shearing stress for the shell wall may be closely approximated as the critical torsional shear in a cylindrical shell whose radius-to-thickness ratio is the same as that of the shell considered and whose length-to-radius ratio is such that tan α as determined from Fig. 5 is equal to r/L. This ratio can be determined quite readily from Fig. 5. In the case of a pipe or tank supported as a simple beam, if the half span is less than the length corresponding to this ratio, the half length might reasonably be used as the effective length of cylinder, since there is no shearing stress at the midspan. For cantilever beams, the full length of the cantilever should be used unless it is greater than the effective length.

A typical example of the cantilever beam action is a stressed-skin fuselage of an airplane. The structural action of such a thin sheet is simulated by the Lundquist tests.[19] The data reported by Mr. Lundquist indicate that, when the bending moment is appreciable, the resistance to buckling in combined shear and bending is less than when the bending moment is small.

A study of these data indicates that, under combined bending and torque, a reliable value of τ'_c for combined shear and bending can be found from the following relation between critical shearing stress (for pure torque) τ_c and the ratio $\frac{\bar{M}}{r\,V}$, by the equation:

$$\tau'_c = \tau_c \left[1 - 0.075 \left(\frac{\bar{M}}{r\,V} - 1 \right) \left(1 + \tfrac{1}{2} \frac{r^2}{L^2} \right) \right] \qquad (33)$$

in which $\bar{M}$ is the total bending moment on the shell, in inch-pounds; and V is the total shear on the shell, in pounds. For values of $\frac{\bar{M}}{r\,V}$ of less than one, τ'_c may be taken equal to τ_c. For values of $\frac{\bar{M}}{r\,V}$ greater than 5, this approximation appears to be ultraconservative. Values of critical shear computed in this way and experimental values obtained by Mr. Lundquist are compared in Fig. 8. Until a more adequate theory is provided such an approximation may be very useful.

SHELLS CONTINUOUS OVER SEVERAL SUPPORTS

A specific example of continuous beam action is afforded by a pipe line filled with water and supported at intervals.[20] If the circular section of the shell is held fixed at the supports by ring girders, or by any other method, the distribution of shears in the side wall is known quite definitely and may be expressed as

[20] *Transactions*, ASCE, Vol. 98, 1933, p. 154.

follows for either a single span or one of a long series of equal spans:

$$S_{ys} = -q\,r\,\frac{L}{2}\sin u = \tau\,t \qquad (34)$$

As would be expected, the shear is a maximum at the ends of horizontal diameters and decreases very slowly for small angular distances from this point. Hence, it may be assumed to be practically constant for about one quadrant of arc. The critical shearing stress in the side walls in this case will also depend upon the ratio $\frac{\bar{M}}{r\,V}$ at the supports and the value of τ'_c determined from Eq. 33, using a value of τ_c determined for a length L equal to $\frac{\pi\,r}{2}$.

Conclusions

A review of the foregoing analytical study of perfectly round cylindrical shells and shells departing slightly from perfection, the comparison of the results of this analysis with experimental results, and the adaptation of the analysis to other practical problems suggest the following conclusions:

1. The analysis has yielded a single equation (Eq. 22*b*) which together with the accompanying curve chart (Fig. 5) provides a procedure for computing a value for the critical shearing stress, the number of wrinkles, and the spiral angle of the wrinkles for thin-walled cylindrical shells of any proportions subjected to pure torsional stresses.
2. Slight departures from perfect roundness cause considerable decrease in the computed critical torsional shearing stress for thin-walled cylinders.
3. Comparison between measured and computed critical shearing stress values indicates agreement within the limits of accuracy consistent with the known variables.
4. The analytical results may be used to estimate values of critical shear for stiffened cylinders and cylinders subject to transverse shear and bending.
5. Comparisons between measured and computed values of critical shear in transverse shear and bending indicate that the computed estimates are sufficiently close to the measured values to serve as a basis for design.
6. Additional test data are needed to determine the reliability of the estimated values of critical shear for cylinders reinforced by longitudinal and transverse stiffeners.

Acknowledgments

Most of the work of preparing this paper was done while the writer was an employee of the Aluminum Research Laboratories of the Aluminum Company of America and had at his disposal the original data of experimental work performed under the direction of Mr. Templin, the assistant director of research.

The writer wishes to thank Messrs. R. L. Templin and E. C. Hartmann, Members, ASCE, and R. L. Moore and Marshall Holt, Assoc. Members, ASCE, all of the Aluminum Research Laboratories, for their constructive suggestions during the development of this paper. Among the drawings and photographs in the paper, Fig. 3 is included by courtesy of the Aluminum Company of America. The essential parts of this paper were first presented as a thesis to the University of Nebraska at Lincoln, in 1938, in partial fulfilment of the requirements for the professional degree of Civil Engineer.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

REPORTS

WATER SUPPLY ENGINEERING

REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION FOR THE TWO YEARS ENDING DECEMBER 31, 1946

The events of greatest influence on water supply as well as on all other activity in the past two years were the surrender of the Germans in World War II in May, 1945, and that of the Japanese in August, 1945. Soon thereafter most manufacture of war goods was stopped; but controls of material were not relaxed until late in 1946. More or less futile attempts to divert the energies of production to housing for veterans, together with strikes in the basic industries, have greatly hindered a resumption of normal activities in water supply improvement and extension, as in everything else. Allotment of cast iron is still under government control.

Some state governments made grants of money to municipalities for postwar planning; but made-work was not necessary in the face of the great dearth of consumer goods. The tendency to look to the federal government for aid, as in the era of the Public Works Administration (P.W.A.) and that of the Federal Works Administration (F.W.A.), appears to have abated considerably as indicated by a canvass of states made late in 1944.

It is not clear what is becoming of all the water supply goods that are produced. Many ordinary projects that one knows about are delayed for lack of material and yet production, apparently greater than that before the war, seems to be already absorbed for nearly a year ahead. Quotations on deliveries of pipe of all kinds seldom specify less than nine months and often much more. Steel tank deliveries are similar. Small direct-connected pumping units appear to be obtainable at reasonable deliveries but pumps, motors, and controls of ordinary and large sizes are quoted at delivery in nine months to a year or more.

Contract prices have soared following the law of supply and demand, augmented by the spiral of inflated labor and material costs. The latter were caused by industry-wide strikes and by the resultant uncertainty as to future prices and wages in the coal, steel, transportation, and machinery industries. So-called escalator clauses are common for materials and equipment. Not a few price quotations are merely statements that goods will be billed at the prices current at time of shipment, which cannot be for months in the future.

NOTE.—Please forward all comments on this report directly to Chairman Thomas H. Wiggin, 90 Broad St., New York 4, N. Y.

Engineers and management are nevertheless accepting these prices and letting contracts in spite of them. The low interest rate on borrowed money—apparently a product of federal government manipulation—has made high prices less of a bar to the undertaking of improvements. (The rate is approximately $1\frac{1}{2}\%$ for good municipal risks and about double that for privately-owned utilities. Revenues from the latter are subject to heavy taxation.) The effect on savings of all kinds has been disastrous.

Production of Cast-Iron Pipe and Cement Lining Therefor

Cast-iron pipe foundries have kindly furnished considerable data as to their production (see Table 1). All mention shortage of raw materials and some mention labor troubles as having reduced production. An estimate of about 70% of output for 1941 was stated by one large producer. In general, 1941 was a high production year. Another large foundry reports many difficulties but with production approaching the high years of 1941 and 1942. Another large foundry reports production about equal to that of 1939 and 1940 with facilities for 25% greater production unusable because of lack of raw materials. This foundry has orders not only for the 1947 output but also for the entire 1948 output, at present rate of raw material supply, and hopes for more material. Still another foundry reports a record production of 2-in. cast-iron pipe but with a general curtailment on account of a shortage of raw materials. This information appears to indicate a fairly high rate of production as compared with prewar production but an inability to apply the increased facilities provided for war needs to the large unsatisfied demand because of shortages caused by labor troubles.

TABLE 1.—Lining and Coating for Cast-Iron Pipe

Year	Percentage of Total Output (100%)			Bituminous coating over cement lining[a] (%)
	Cement lining of standard thickness	Thin cement lining	Tar-dipped cast-iron pipe	
(1)	(2)	(3)	(4)	(5)
Foundry A:				
1945.....	23	23	54	95
1946.....	26	26	48	95
Foundry B1:				
1945.....	10	21	69	0
1946.....	12	30	58	0
Foundry B2:				
1945.....	1	4	95	0
1946.....	1	4	95	0
Foundry C:				
1945.....	26		74	..
1946.....	12	27	61	..[b]
Foundry D (4-In. to 12-In. Pipe):				
1945.....	12	0	88	75
1946.....	18	0	82	75
Foundry D (Pipe Smaller Than 4 In.):				
1945.....	26	0	74	75
1946.....	33	0	67	75
Foundry E1 (3-In. to 12-In. Pipe):				
1946.....	6.5	63.8	10.3	..[c]
Foundry E2 (12-In. to 48-In. Pipe):				
1946.....	22.7	40.4	36.9	..[c]
Foundry F:				
1945.....	23.6	0[d]	..	100
1946[e].....	24.5	0[d]	..	100

[a] Percentage of pipe lined with cement—that is, percentage of the pipe represented in Cols. 2 and 3. [b] 50% of the standard thickness (Col. 2) and all the thin cement lining (Col. 3). [c] 25% of the standard thickness and all the thin cement lining. [d] Foundry F stated that it does not believe in the thin lining. [e] Eleven months.

The committee wishes to express its satisfaction in the growth in use of cement lining but again expresses its doubt as to the wisdom of using the thin linings.

Reinforced Pressure Pipe

Reinforced concrete pressure pipe continues to be used extensively even in smaller sizes—at least to 16 in. This is particularly true of pre-stressed pipe

in which wire of high elastic limit and high ultimate strength is used for the major reinforcement. The number of firms engaged has increased. One firm reports sales, in 1945, of about 50 miles of 20-in. pipe to 84-in. pipe, about one fourth of which was of the pre-stressed type. Sales in 1946 were much reduced but there is a very large backlog of orders both for the United States and for South America so that quotations on deliveries are for months ahead. Practical methods of making wet connections have been developed.

Lining of Large Pipe in Place

The process first developed, in which premixed mortar is spattered on to the pipe by centrifugal force and smoothed by revolving trowels, has continued in extensive use. During 1945 and 1946, to December 3, 1946, a total of nearly 348,000 ft in sizes from 30 in. to 48 in. was reported as being lined by this company. Another company, using a revolving gunite nozzle and revolving trowels, has done one or two jobs. Gunite applied by hand nozzle and smoothed by hand trowels has also been used for at least one pipe line of considerable length.

Lining of Small Pipe in Place

The process of lining pipe varying in diameter from 3 in. to 16 in. with cement mortar by pulling a mandrel through the pipe (used so extensively in other countries) has reached a total output of more than 662,000 lin ft in the United States, not including some work in November and December, 1946, not reported. The total for 1945 was more than 123,000 and that for 1946 was more than 133,000 plus some November and December work not reported.

Treatment of Pipe After Cleaning

The well-known drawback to cleaning pipe—its tendency to tuberculate more rapidly after cleaning—has been the subject of chemical research which has not been very successful; and also of invention with respect to coating in place after cleaning. A process, patented many years ago, of applying asphalt from an emulsion with the aid of an electric current has been used to some extent in England but not in the United States so far as the committee knows. An intention to engage in this business soon in America has recently come to the attention of the committee.

Notable Water Supply Projects

New York, N. Y.—Work of the Board of Water Supply of the City of New York on the new Rondout Creek-Neversink-East Branch of the Delaware project for adding 540 mgd to the water supply was interrupted by the war since the War Production Board (W.P.B.) judged that, with care, the city could do without the increased supply. At the time of work stoppage, the Delaware Aqueduct had been constructed from Rondout Creek to West Branch Reservoir on the Croton Watershed; thence to the 30,000,000,000-gal Kensico Reservoir about 40 miles from the city; thence to the 1,000,000,000-gal Hillview Reservoir at the north border of the city; and thence by tunnel across the East River near Hell Gate into Queens County on Long Island, including also a connection

with the older Catskill pressure tunnel which traverses the boroughs of Bronx and Manhattan and terminates in Brooklyn.

Transportation of the approximately 100 mgd of additional supply from Rondout Creek was thus provided but the Merriman Dam on Rondout Creek was only partly constructed and no considerable storage of water was possible. The new aqueduct also was not equipped with its permanent control valves and unwatering equipment. The old contracts had to be liquidated.

During the past two years the natural flow of Rondout Creek has been used. New contracts have been let at enormously increased prices for the completion of Merriman Dam, for the equipment of the Delaware Aqueduct tunnel from Rondout to Hillview Reservoir, and for the caisson cutoffs of Neversink Reservoir farther to the west. The contract for the dam contains a special kind of reverse-escalator clause under the terms of which accurate cost records are being kept and the contractor has engaged to refund to the city 80% of any profit in excess of 15%.

Water consumption of the city has risen greatly and is now about 1,050 mgd, about 200 mgd greater than the low consumption reached during a part of the war years. The exceptionally dry year of 1946 is causing some anxiety as to the adequacy of the supply, which is deficient in accordance with accepted methods of computing dry weather yield.

Boston, Mass.—During the past year the huge Quabbin Reservoir was filled to the level of the spillway. The filling was begun on August 7, 1939, with the closing of the diversion tunnel gates at Winsor Dam. The water reached the sill of the depressed stop log section of the spillway on May 30, 1946. The filling was celebrated on June 22, by lifting the logs on the depressed section of the spillway 30 ft long. Since that time the spillway has been discharging at rates varying from about 70,000,000 gal daily to 20,000,000 gal daily. At the level of the main spillway which is 2 ft above the stop long sill, the reservoir holds about 415,000,000,000 gal available to the district above the sill at Quabbin Aqueduct, including estimated ground storage. During the filling 95,660,000,000 gal have been drawn to date through the aqueduct to replenish Wachusett Aqueduct. Some consideration is being given to supplying other communities near the center of state with water from this reservoir.

The installation of a water wheel at Winsor Dam to generate electric power from the required releases and from waste water was delayed on account of World War II but is now in progress.

Extension of the district's Hultman Aqueduct, to the main center of distribution, was delayed because of the war, but the contract was executed on November 25, 1946, for sinking the necessary construction shafts for the city tunnel section of this aqueduct which will extend about 5½ miles from the Charles River to a connection with the southern high-service lines at Chestnut Hill.

The district has put into effect this year a substantial reduction of approximately 50% in the wholesale rate which it charges its twenty cities and towns for water. Recent legislation has established a flat rate of $40 per million gallons. This was made possible by issuing annually (beginning in 1946) a series of so-called Water Use Development Bonds. In effect, these are re-

financing bonds, which anticipate an expected drop in average cost and spread it as a flat rate over the next twenty-five or thirty years to encourage new municipalities to join the district by making it financially advantageous to abandon their local supplies.

The Metropolitan District Water Supply Commission has been given the job of extending metropolitan sewage works as well as water works, and is now actively engaged in the construction of a sewage treatment plant at the outlet of the South Metropolitan Trunk Sewer in Quincy, Mass.

Detroit, Mich.—In 1946, Detroit filed plans with the State Planning Commission for laying some 25 miles of 36-in. to 54-in. water mains in various locations, at an estimated construction cost of nearly $5,000,000. Other water works improvements are contemplated, estimated to cost about $3,000,000.

Saginaw-Midland Project in Michigan.—This water supply project is designed to deliver 43 mgd from Lake Huron to Saginaw and Midland and to the Dow Chemical Company. The present supply for these cities is from a river heavily polluted by industrial wastes and the water cannot be treated successfully. The project includes a 66-in. steel intake extending 2 miles out into Lake Huron with a wooden crib in 50 ft of water. The intake capacity is 100 mgd.

A pre-stressed reinforced concrete pipe line, 48 miles long, designed for a 300-ft working pressure, is to deliver the water to the Junction pumping station. From this station two 36-in. lines are extended, each approximately 15 miles in length, one going to Saginaw and the other to the Dow Chemical Company in Midland. The two pumping stations have an installation aggregating, at each station, a total of 70-mgd electric-driven pumps. Plans provide for the ultimate addition of two more pumping stations similar to the first two. The facilities being constructed are designed to deliver up to a maximum of 71 mgd—the anticipated peak day requirement with an average use of 43 mgd. All contracts have been let for the construction, the cost of which totals $10,300,000. The letting was on a fixed-price basis with the exception of the $200,000 for pumps and electrical equipment which includes escalator clauses with a maximum provision of 20%. Construction will start early in 1947 with anticipated operation beginning before July 1, 1948.

Chicago (Ill.) Filter Plants.—A start was made on the planning of addition filtration plants for Chicago estimated to cost from $55,000,000 to $60,000,000.

Construction work continued throughout 1946 on the South District filtration plant in Chicago, which, when completed, will cost about $24,000,000. Based on the customary rating of 2 gal per sq ft per min for filters, the filter capacity of the plant is 320 mgd. The plant is designed for a maximum average daily load of 450,000,000 gal and a maximum hourly load of 600,000,000 gal. Construction on this project proceeded slowly during the war period with priorities granted for completion of certain parts of the plant. If conditions continue as at present, another year of construction will be required.

The South District filtration plant was placed in partial operation on August 16, 1945, first with intermittent daily use; and, on October 8, 1945, continuous operation with partial treatment of the water was begun. The partial treatment involves low-lift pumpage of the water, coagulation, and sedimentation.

In the early part of 1947 a few of the filters will be ready for operation. All the filters are expected to be in operation by July or August, 1947.

The use of activated carbon will be started as soon as the water from one section of the plant can be filtered. A small amount of lime is being used to reduce the corrosiveness of the water. The sodium silicate treatment for aiding coagulation is being tuned up. The difficulty in obtaining silicate and sulfuric acid has delayed the use of this treatment.

Louisville, Ky.—The city-owned Louisville Water Company is planning to change from steam to electric drive for the Crescent Hill high-service pumping station. The 155-mgd steam units are to be retained for reserve and 140-mgd electric-driven equipment is to be installed. It is estimated that the electric supply will be more economical than the continuation of the steam power.

Des Moines, Iowa.—Plans and specifications have been completed for a 36-mgd iron removal and softening plant at Des Moines. This construction will be undertaken when prices are more favorable.

St. Louis, Mo.—Plans and specifications have been prepared for a $19,000,000 project to expand and improve the water purification system of St. Louis. This program includes also the improvement of the distribution facilities for an increase of 20 mgd in the output. The consumption averages about 150 mgd. A $7,500,000 bond issue for some of this work was authorized at a public election on August 1, 1944. Other necessary funds were already authorized or in hand.

Kansas City, Mo.—Improvements to the Turkey Creek steam pumping station are under construction and a new feeder main for western part of Kansas City, Mo., has been designed.

Kansas City, Kans.—Equipment has been purchased for improvements to the water treatment plant at Kansas City, Kans., which is to be rebuilt when construction prices stabilize.

Omaha, Nebr.—A new river intake, new mixing basins, and feeder main additions to the distribution system are planned for Omaha.

Oklahoma City, Okla.—The construction of the Bluff Creek Filtration Plant and Pump Station was begun in May, 1945, and has been completed recently. This plant was designed for an initial capacity of 15 mgd and arranged to be ultimately extended to 60 mgd.

The water is taken from Bluff Creek about 8 miles northwest of the city and is part of the new water supply system that has been constructed with a bond issue of $6,911,000 voted in 1939. The dam, canal, and water main extensions were completed in 1944, leaving the filtration and pumping station as the remainder of the plant unfinished.

The plant provides for prechlorination and postchlorination if necessary and for the use of lime, alum, and soda ash with an underwater carbon dioxide gas burner and the application of hexametaphosphate after filtration. The filters are provided with a rotary surface wash system.

Savannah, Ga.—At present Savannah, which obtains its water supply from ground water, has become afraid of encroachment of salt water at the well field where the principal aquifer is the Ocala limestone, and has decided to secure an additional supply from Abercorn Creek, a tributary of the Savannah

River. The original plant is to have a capacity of 35 mgd with provisions for readily expanding the capacity to 50 mgd. The raw water will have to be pumped to a filtration plant to be located about 8 miles from the Abercorn Creek pump house.

An unusual feature of the filter plant is the use of liquid alum delivered in either trucks or tank cars from a new liquid alum plant in Savannah. Hydrated lime is also to be used, together with prechlorination and postchlorination. The filters will have a nominal capacity of 40 mgd.

The new supply is to be used mainly for industries and is expected to have a hardness of less than 1 grain per gal. The water, however, will be available to supplement the regular present domestic supply. If possible, it is planned to have the new water supply available toward the end of 1947. A 48-in. main about $1\frac{1}{2}$ miles long is to deliver the treated water to the present system.

Atlanta, Ga.—A water works expansion program looking forward to the next twenty-five years has been developed by Atlanta. The program includes a new filter plant having a capacity of from 15 mgd to 18 mgd designed so that it could be increased later to 54 mgd. A number of large connecting mains are to be installed, including 11 miles of large mains in the distribution system.

Houston, Tex.—In July, 1943, the city council of Houston adopted an official long-range plan for water improvements including the following, estimated to cost $30,000,000: Expand ground-water sources to 75 mgd; develop a surface water supply of 135 mgd; and provide necessary pumping stations, treatment plant, transmission, and primary, secondary, and grid mains.

In 1944, $14,000,000 were given to the immediate program as follows: Additional wells providing 18 mgd; a high-lift pumping station, 50 mgd; suction storage, 10,000,000 gal; large mains, 36 miles; small mains, 50 miles; a dam on San Jacinto River, 20 miles northeast of Houston; a low-lift pumping station; a transmission canal, 200 mgd ultimate; a treatment plant and storage for the first unit, 25 mgd; a high-lift station, first unit, 75 mgd; and six miles of transmission mains, 30 in. to 36 in.

The greater part of the ground-water developments, including the mains connected to them, has been completed; but the major part of the surface supply development is in the planning stage, although the transmission canal and temporary low-lift pumping station were acquired from F.W.A. Bids for the dam received in October, 1946, were rejected because they were too high. A pilot treatment plant has been operated since March, 1946, and is to be continued for one year. The bid for the dam was $7,500,000 compared with an estimated cost of $5,200,000 at present index prices. At the end of 1946 somewhat more than $6,000,000 had been expended or appropriated for uncompleted contracts then in force. The metropolitan area of Houston uses about 118 mgd which is mainly from the ground-water source of supply. The municipal water system furnishes about 50 mgd of the total demand.

Metropolitan Water District of Southern California.—Plans and specifications are being prepared for a second unit of the softening and filtration plant. This addition will double the present capacity, raising it from 100 mgd to 200 mgd. It is contemplated that the construction specifications will be issued early in 1947.

Julian Hinds, M. ASCE, advises that, in the fiscal year 1945–1946, the demand for water from the Colorado River Aqueduct was 151% of the demand in 1944–1945. The peak demand during the summer of 1946 was 104 mgd as compared with the rated capacity of the filtration plant at 100 mgd.

With the recent annexation of the San Diego County Water Authority to the Metropolitan District, water for the San Diego Aqueduct will be supplied from the Colorado River Aqueduct.

San Diego, Calif.—The San Diego County Water Authority, which includes the City of San Diego and six other cities and districts in the vicinity, has voted, by an overwhelming majority, to annex to the Metropolitan Water District of Southern California. The United States Navy is constructing an aqueduct from a point near the west portal of the San Jacinto Tunnel of the Metropolitan Water District a distance of approximately 70 miles to the San Vicente Reservoir, one of the main storage facilities of the City of San Diego. This line is estimated to cost $15,000,000 and is about 50% complete. The estimated date of completion is December 1, 1947. This aqueduct, although being installed by the U. S. Navy, is to be paid for by the San Diego County Water Authority. It will have a capacity of 50 mgd.

The San Diego County Water Authority is designing pipe lines to distribute water from the aqueduct being constructed by the Navy to the other member agencies of the authority outside San Diego. Bonds for this purpose were voted at the November 5 election in the amount of $2,000,000.

A contract has been let for the construction of 12 miles of pipe line extending from San Vicente Reservoir, the terminus of the San Diego Aqueduct, to the San Diego distribution system. This pipe line will permit bringing the Colorado River water into the City of San Diego. The lines will have a length of some 12 miles in which the pipe will range from 48 in. to 60 in. in inside diameter. The estimated cost of this work is $4,000,000.

Plans are being prepared for the construction of the Alvarado filtration plant in East San Diego with a design capacity of 66 mgd and an ultimate capacity of 100 mgd. This plant will be designed to handle filtration and softening of local and Colorado River waters, using the lime zeolite process for softening, with rapid sand filters. The estimated cost is $2,250,000.

The City of San Diego has under way, or in the planning stage, a large amount of additional work required to strengthen the distribution system. A considerable amount of additional pipe lines, together with distribution tanks, reservoirs, and pumping plants, is involved. An expenditure of approximately $3,000,000 is contemplated during the next two years on these features. It is reported that the entire distribution system is badly overloaded because of the tremendous increase in population as the result of the war. Additional studies are being conducted to provide for the further development of waters on the various streams in San Diego County, with particular attention to the San Dieguito River.

Los Angeles, Calif.—Plans and specifications are completed and work will begin about January 1, 1947, on the construction of a distribution reservoir with a capacity of 900 acre-feet. The estimated cost is $4,500,000. The reservoir will have an inlet-outlet line 80,000 ft long, composed of large-

diameter steel pipe. This reservoir is planned to improve the distribution of the supply in the southwest part of the city.

Three desilting structures along the open section of the Owens Valley Aqueduct between Haiwee and Alabama Hills are to be constructed in the near future. The estimated cost is $36,000. Two of these structures will desilt the stream waters from side creeks and permit clear water to be discharged into the aqueduct. The third is to be built in the aqueduct itself, at the Alabama Hills gate, to desilt the aqueduct water and to permit the debris to be discharged at the Alabama Hills spillway gate, thus providing clear water between that point and the Haiwee Reservoir.

Three reinforced concrete overhead spillways across the open section of the Owens Valley Aqueduct between the intake and Alabama Hills are also planned. Along with this work there will be some remodeling of the existing spillways. The total estimated cost is $125,000.

Researches by Los Angeles Water Dept.—

Pipe Joints.—Experiments have been in progress in connection with the joining of small-diameter cement-lined standard steel pipe by the use of welded butt straps, both with and without water in the pipe. The heat of welding does not injure the cement lining. In this connection taps have also been made to the pipe by the use of couplings welded on with no damage to the cement lining. The department advises that the use of welded butt straps on cement-lined steel pipe has been very helpful because it provides rigid joints for fairly long spans between supports.

Clarification of Stored Water.—Research work is in progress on the clarification of stored surface water supplies. In the investigation under way, experiments are being made with the diatomaceous earth filtration as developed by the United States Army during the war. Preliminary results indicate considerable saving in installation and operating costs as compared with the standard design of filtration plants. Water with a turbidity of less than 0.1 ppm can be readily obtained. It is anticipated that water with such a low turbidity would probably eliminate many complaints of odor and tastes resulting from bacteriological slime growths in pipe distribution systems receiving unfiltered stored surface water.

Electron Microscope.—The management of the water system has recently installed an electron microscope to investigate and identify bacteria having sanitary significance, and also to conduct research on the chemical and microscopic analyses of water.

Chemical Weed Control.—The Water Department of Los Angeles has made important additions to the researches of others in control of land and water weeds in and near canals and reservoirs. The conclusions of R. F. Goudey, M. ASCE, are stated by him briefly as follows:[1]

> "Land and water weeds can be economically controlled by the use of chemicals, provided such work is properly supervised and intelligently applied.

[1] "Chemical Weed Control," by R. F. Goudey, *Journal*, A.W.W.A., February, 1946, pp. 186–202.

"Land weeds and water emergent weeds are generally best controlled by the new 2,4-D compounds, but other chemicals have specific advantages for different purposes.

"Submerged aquatic weeds are generally best controlled with chlorinated hydrocarbons, employing higher doses than previously recommended (up to 50 gal. per acre), allowing much longer contact periods (up to four days) than formerly, and by observing necessary factors to avoid complaints of odors and tastes from consumers."

Litigation Re Owens Lake.—Many cases have been tried on different phases of this diversion. The case here noted is the last. It was decided in 1943 but the effects are still being evaluated. The peculiarity is that the City of Los Angeles was sued by Natural Soda Products Company for damage which it was claimed could have been prevented if the city had diverted flood-waters out of the valley or disposed of them by spreading. Decision was adverse to the city and the decision places such a burden on water "appropriators" in California that a concerted movement is asked to obtain reconsideration and reversal.

San Francisco, Calif.—San Francisco has under construction a 35,000,000-gal distribution reservoir known as Sutro Reservoir at an elevation of 500 ft above sea level on the slopes of Twin Peaks.

A new 60-in. steel transmission line is being constructed from San Andreas Reservoir to Sunset Reservoir in the distribution system, a distance of approximately 11 miles.

Plans are being developed for the construction of a second pipe line on the Hetch Hetchy Aqueduct across the San Joaquin Valley and of a third line from the Irvington Portal of the Hetch Hetchy Aqueduct across the south end of San Francisco Bay to the Pulgas Tunnel. Preliminary plans are also being developed for a filtration plant below the Crystal Springs Dam. At present all San Francisco water is delivered without filtration.

East Bay Municipal Utility District (Supplying Berkeley, Calif., and Neighboring Territory).—On November 5, 1946, a bond issue of $12,000,000 was voted by the residents of the district to provide funds for the construction of a second pipe line on the Mokelumne Aqueduct extending from Pardee Tunnel to Walnut Creek Tunnel, a distance of 81 miles. This pipe line will have an inside diameter of 67 in. and will be under a maximum head in crossing the delta of approximately 500 ft. It is estimated to cost approximately $22,000,000 and the remainder of the funds required for construction will be provided from reserves and from revenue from water sales. Specifications have already been issued for the installation of 30 miles of this aqueduct extending eastward from the Walnut Creek Tunnel. The specifications provide for obtaining bids on steel pipe, cement lined on the inside and gunited on the outside, and also (as an alternative) on pre-cast concrete pipe with a steel cylinder. The capacity of this second pipe line will be 50 mgd by gravity, which can be increased ultimately to slightly more than 100 mgd by installation of two booster pumping plants. The present aqueduct has a capacity of 95 mgd with pumping at the Walnut Creek and Bixler pumping plants.

Work is in progress on the enlargement of the San Pablo and Orinda filter plants. The San Pablo plant is being enlarged from a present capacity of 36

mgd to a proposed capacity of 54 mgd. The Orinda plant is being enlarged from the present capacity of 42 mgd to the proposed capacity of 105 mgd. Contracts have been let for both jobs and work is in progress. It is anticipated that both plants as enlarged will be in service during 1947.

The district is constructing five pre-stressed concrete storage tanks in the distribution system, with capacities varying from 1 mgd to 3 mgd. These storage tanks follow the same design as described by officers of the district in various articles which have appeared in engineering magazines during the past few years. The contract cost for these five tanks is $388,464.

In December, 1945, the district completed the construction of the Bixler Pumping Plant on the Mokelumne Aqueduct, increasing the capacity of the existing line from 67 mgd to 95 mgd. This plant is situated in the delta section and consists of two 47.5-mgd pumps, each driven by a 4,000-hp motor. At this plant the lift is approximately 450 ft. A description of the general layout and the operation of the plant has been presented by H. A. Knudsen.[2]

In addition to the foregoing principal items, the district has done a large amount of work to improve and extend the distribution system. This has involved, during the past year, the laying of some 39 miles of pipe, the construction of a number of additional distribution pumping plants, and the enlargement of a number of existing pumping plants.

Long Beach, Calif.—Engineers for the City of Long Beach are preparing plans for a new water treatment plant which will remove the color and provide filtration. Water for Long Beach is obtained from wells and also from the Colorado River Aqueduct.

Santa Barbara, Calif.—This community is planning the enlargement of its existing storage on the Santa Ynez River or the construction of a new reservoir.

Monterey, Pacific Grove, Carmel, Calif.—The California Water and Telephone Company, which serves these and other settlements on the Monterey Peninsula, obtains its water by diversion from the Carmel River. This company is planning the construction of a storage dam on the Carmel River to provide an additional supply to take care of the large increase in consumption. A filter plant is also being designed for installation near the existing diversion dam.

Contra Costa Canal, California.—The Bureau of Reclamation expects to complete the Contra Costa Canal within the next year, a unit of the Central Valley Project, through to Martinez, Calif. The City of Martinez has developed plans for taking water from the canal and has voted the necessary bonds to provide for the construction of a modern filtration plant.

Federal River Projects

The committee's report for the years 1943 and 1944[3] contained a rather complete analysis of the wording and apparent scope of House Bill 4485, which became Public Law 534, 78th Congress. The act seemed to give the broadest powers to the Corps of Engineers in the development of the river systems of the

[2] "Check Valves End Water-Hammer Risk," by H. A. Knudsen, *Engineering News-Record*, September 5, 1946, pp. 310–312.

[3] *Proceedings*, ASCE, May, 1945, p. 679.

United States for all purposes, and authorized to be appropriated more than $1,000,000,000 with which to proceed. The committee has sought the aid of an important officer of the Corps in an attempt to understand and express what the Corps regards as its powers and duties under House Bill 4485 (Public Law 534) and other legislation. Some direct and indirect quotations from the information which he has so kindly furnished are:

"The present civil activities of the Corps of Engineers are the result of an historical growth beginning with the infancy of our country when military engineers were the only professionally trained engineers available. There is no single basic act of Congress which defines these activities.

"The functions of the Corps of Engineers are divided between military operations and civil works. The former are broad in scope and are largely controlled by military personnel exercising authority delegated by the President as Commander-in-Chief of the Army. These activities are carried on in peace-time periods on a curtailed basis. The carry-over includes the training of Engineer officers and troops both in a small Regular Army and in the reserves and the National Guard, and the organization of engineer supply including the development of special equipment and the prosecution of military mapping.

"The civil activities of the Corps of Engineers, which originally were assigned under executive authority of the President, have since been formalized by specific acts of Congress, which delegate to the Sectetary of War and to the Corps of Engineers, the duties conferred on the Federal Government under the Constitution of improving, maintaining and controlling the navigable waters of the United States. This delegation of authority is neither comprehensive in extent nor basic in law. It derives largely from the accumulation of laws known loosely as the 'River and Harbor' and 'Flood Control' Acts which are passed almost yearly by the Congress. Many exceptions, however, to Corps of Engineers control of our navigable waters may be found, of which perhaps most notable is the law establishing the Tennessee Valley Authority.

"The principal Congressional Acts defining these activities are as follows:

"*a.* The River and Harbor Act of 1892 was the first to specify generally that river and harbor works were to be performed under the supervision of the Corps of Engineers.

"*b.* The Act approved 13 June 1902 created the Board of Engineers for Rivers and Harbors, a board of Engineer officers established to review projects prior to their submission to Congress through the Chief of Engineers. This Act, together with that of 4 March 1913, established the basic procedures under which navigation reports of the Corps of Engineers are prepared, reviewed and transmitted to Congress for possible enactment into law.

"*c.* The Act of 1 March 1917 extended to flood control projects, the same procedure for authorization as was specified by prior Acts for navigation projects.

"*d.* The River and Harbor Act approved 21 January 1927, in accordance with House Doc. 308, 69th Congress, 1st Session, and the Flood Control Act approved 15 May 1928 (Sec. 10) authorized a comprehensive study of all waterways with respect to navigation, water-power development and related water uses, such as flood control, domestic or industrial water-supply, irrigation, sanitation, pollution abatement, wild-life conservation and recreation, but did not direct that these subjects were all necessarily a Federal

responsibility. These investigations were extended, under the authority of the Chief of Engineers, by the River and Harbor Act approved 30 August 1935.

"*e.* The Flood Control Act approved 22 June 1936 contains the first broad declaration of policy on flood control enunciated by Congress. This was modified and expanded by subsequent Acts, particularly those of 28 June 1938 and 22 December 1944, which define the Federal interest and responsibilities for improvements for flood control and allied purposes.

"*f.* The River and Harbor Act approved 3 July 1930 provided for investigation of beach erosion problems by the Corps of Engineers in cooperation with local governments; and the Act of 31 July 1945 established a policy of Federal aid for the construction of shore protection works."

With respect to Public Law 534, this has had as yet little impact on the work of the Corps. The typical procedure remains as before and is as follows:

"*a.* An item is included in a River and Harbor or Flood Control Act directing a preliminary examination and survey of a particular waterway.

"*b.* This is assigned to the appropriate District Engineer, who holds a public hearing to obtain the desires of local interests in the project, and who then prepares a preliminary examination report with a view to deciding whether or not the project warrants detailed study.

"*c.* The preliminary report is reviewed by the Division Engineer, the Board of Engineers and the Chief of Engineers. If it is negative, it is returned to Congress and no further action is taken unless a new authorization from Congress is received. If the report is favorable, it is returned to the District Engineer for preparation of a survey report which includes a detailed study and estimate of cost.

"*d.* The survey report is reviewed through the same chain of command as the preliminary report. On a major project, a public hearing may be held by the Board of Engineers to obtain first-hand information on the need for the project. Under the Act approved 22 December 1944 (Public Law 534—78th Congress, previously referred to) the report is also submitted by the Chief of Engineers to the Governors of the affected States prior to submission to Congress. When the report reaches Congress it goes to the appropriate standing committee of the House who may draft legislation to authorize the project. Here again public hearings may be held. Action thereafter follows normal legislative procedure.

"*e.* After the project has been authorized, the Chief of Engineers programs the work and submits an annual estimate to the Budget Director who prepares the national budget for the President. A subsequent appropriation bill is then introduced in Congress for necessary funds."

The committee has had some indirect contacts with a few works listed in Public Law 534 and knows of a hearing at which local municipalities or states were asked their desires regarding participation financially. It so happened that in these cases no approved plans have emerged yet; the fact that these plans contemplated that much of the cost would be borne by the localities benefited was an important deterrent.

Public Law 534 contains the provision:

"Sec. 10. That the following works for the benefit of navigation * * * and other purposes are hereby adopted and authorized [these were briefly listed in the committee's previous report] * * * with a view toward providing an adequate reservoir of useful and worthy public works for the post-war construction program. * * * Provided, that the necessary plans, specifications, and preliminary work may be prosecuted on any project authorized in this Act to be constructed by the War Dept. during the war, with funds from appropriations heretofore or hereafter made for flood control, so as to be ready for rapid inauguration of a post-war program of construction; provided further, that when the existing critical situation with respect to materials, equipment, and man power no longer exists, and in any event not later than immediately following the cessation of hostilities in the present war, the projects herein shall be initiated as expeditiously and prosecuted as vigorously as may be consistent with budgetary requirements."

Much power still exists, apparently. The committee's statement in the 1943–1944 report still appears to be pertinent,[4] "River work which is likely to take a generation or more has been assigned to the War Department by Congress * * *." Budgetary requirements and the very magnitude of the task will apparently make for slow progress unless a depression should occur.

Another contemplated assignment to federal agencies of jurisdiction or partial jurisdiction over interstate waters and their tributaries is in the matter of stream pollution. Since 1935 federal legislation for the control of stream pollution has been before Congress, but no definite action has been taken. Groups desiring the protection of wild life have favored legislation that would place stream pollution control under federal jurisdiction including federal court action. Other groups have urged that the states continue their control over stream pollution with federal participation in a fact finding and advisory capacity.

Acting upon the recommendation of the Sanitary Engineering Division of the Society, the Board of Direction presented a statement at a public hearing of the Committee on Rivers and Harbors of the House of Representatives on November 13, 1945, regarding several bills that were under consideration for coordinating the control of stream pollution. The statement in general favored H.R. 4070 and endorsed a comprehensive coordinating program directed by the Sanitary Engineering Division of the United States Public Health Service as provided in that bill.

As a result of conflicting opinion as to the extent to which federal jurisdiction should be provided, no bill was passed and all bills died with the session of 1946.

Various efforts were made by representatives of the different groups interested in federal legislation on stream pollution control to secure some agreement on proposed legislation before the new Congress met at the beginning of 1947. The group that made the most progress was one called together in New York on November 22, by the chairman of the Committee on National Water Policy of the Conference of State Sanitary Engineers. About a dozen persons were present at this meeting representing, respectively, commissions and com-

mittees of the United States Public Health Service on stream sanitation, water works and sewage works associations, the American Public Health Association, the New York State Department of Health, and wild life organizations. An agreement was reached on proposed modifications of H.R. 4070. This compromise draft was submitted to the annual conference of the Association of State and Territorial Health Officers held in Washington on December 2 and 3, 1946, and approved in principle.

Salient provisions in the bill as modified are as follows: The Surgeon General, counseled by an advisory board of nine persons representing federal, state, and municipal governments, sanitary engineering, and industry, would be empowered to notify state or interstate authorities of uncorrected pollution and outline reasonable and equitable measures. After two years of neglect, legal action may be taken by any United States attorney at the request of the Surgeon General.

Provisions are also included for a survey of pollution by the Surgeon General and for certain financial aid by the federal government.

On January 3, 1947, H. R. 315 was introduced in the House of Representatives and is the bill that carries out the aforementioned suggestions. At the same time, another bill, H.R. 123, was introduced which included many of the provisions of H.R. 315, and also includes some provisions for greater federal authority that had been in H.R. 6024 and had been objected to by those who are in favor of H.R. 4070.

It is obvious that federal control of sources of water supply has greatly increased in recent years beyond the original authority for navigation and is likely to increase further. It is the opinion of the Committee on Water Supply Engineering that the Society in general and the Sanitary Engineering Division in particular should continue their interest in these matters. They should strive to prevent too great a concentration of authority and to obtain proper recognition of the competence of civilian engineers and of their right to share on equal footing with the military engineers in the responsibility for these important projects.

Postwar Planning

Public Law 534, 78th Congress, contemplates postwar planning on a large scale—too large apparently to be accomplished without the drive of necessity. The Society's activity in postwar planning has doubtless borne some fruit, but, as predicted in the 1943–1944 report,[4] the need has not developed. Many plans to meet active demands caused by the lag in facilities due to the absorption of materials and man power by the war have been, and are being, developed and are generally waiting for material and labor to be available. Many priorities are still unfulfilled and are being worked on although the priority system is largely abolished.

Protection from Sabotage, Mutual Aid, Cross Connections

These subjects were treated at some length in the 1943–1944 report[5] but how quickly they have ceased to be of major importance! Many connections

[4] *Proceedings*, ASCE, May, 1945, p. 680.
[5] *Ibid.*, p. 681.

made for mutual aid in wartime still exist, however, and can be useful in peacetime. Not a few of these were limited by law to the duration of the war. In some of these cases a change in law may be worth seeking.

The sanitary questions relating to cross connections are still pertinent and the wartime solutions mentioned in the 1943–1944 report are doubtless permanent in many cases. The committee would welcome some correspondence to develop this subject further. A letter on this subject from R. F. Goudey, M. ASCE, is included as an Appendix to this report.

Floods, Hurricanes, Droughts

Although the years 1945 and 1946 did not bring any such startling floods and hurricanes as the years 1936, 1937, 1938, and 1942, the *Water Resources Reviews* issued by the U. S. Geological Survey for 1945 [6] state: "* * * There were notable floods in 10 months of the year * * * and many new records were established." Likewise, for 1946 [7] it states, "* * * intense local floods occurred throughout the year."

Examining the 1945 records more in detail, the February–March flood on the Ohio River was the most damaging, being second in Ohio River flood history only to that of 1937, although 10 ft lower. Record floods occurred in Missouri, Oklahoma, and Arkansas. By far the largest flood on the Meyer formula scale of $X \sqrt{\text{drainage area}}$ was that in Sallisaw Creek, Okla., on March 30, 1945—608 cu ft per sec per sq mile on 181 sq miles, or approximately $8{,}150 \sqrt{A}$. This is a larger rate of runoff on the Meyer scale than that for any flood in the Port Allegany, Pa., storm of July, 1942, which was noted at length in the 1940–1942 report[8] of the committee as a maximum in the United States up to that time.

Although some of the 1946 floods broke records for their locality, no rate of runoff reached a higher figure on the Meyer scale than about $3{,}400 \sqrt{A}$. Illustrating the damage that even a flood of comparatively low peak rate can do where total volume is large and channels are constricted, the flood in the Chemung and near-by river basins in New York and Pennsylvania in May, 1946, flooded the City of Elmira, N. Y., inundated the water supply pumping station, and swept away numerous bridges. This flood had a peak rate of 59 cu ft per sec per sq mile on 2,530 sq miles, corresponding with a Meyer rating of $2{,}500 \sqrt{A}$. The total precipitation averaged from twenty-seven stations was 3.77 in. with a maximum of 7 in. The ground was wet from previous rains.

All these published records of floods have the failing that they omit hydrographs or data to permit the construction of hydrographs.

As to droughts in 1945 the record[9] states: "None were extreme either in areal extent or intensity * * *." Noted as of 1946 but beginning in June of 1945, intense drought occurred in Arizona, New Mexico, Utah and Southern Colorado, reaching its maximum intensity in June 1946. Stream flows set

[6] *Water Resources Review*, U.S.G.S., water year 1945, p. 4.
[7] *Ibid.*, water year 1946, p. 3.
[8] *Proceedings*, March, 1943, p. 413.
[9] *Water Resources Review*, U.S.G.S., water year 1945, p. 6.

new low records and extensive damage was caused to crops insufficiently irrigated.[7]

During the latter half of 1946 and continuing, there has been a marked deficiency of rain in New York State and vicinity which has greatly decreased water supply reservoir storage and has caused emergency well supplies to be used.

Dams and Spillways

Masonry Dams.—No new principles in the design of masonry dams have come to the committee's attention. The committee is glad to note, however, that there is renewed activity, and that conferences of sub-committees on masonry dams have been planned for the January Meeting of the Society. These committees were initiated at the Chattanooga (Tenn.) Meeting, in April, 1939, to further the scanty knowledge on uplift water pressure and ice pressure on masonry dams. Little real knowledge exists on either subject and designs have been based on assumptions mostly dating back nearly fifty years.

Earth Dams.—The construction of dams has been curtailed by war conditions during the past five years, and no important developments in the fundamentals of design or in the methods of construction outlined in earlier reports of this committee can be recorded.

Good practice will continue to depend on the treatment of soil as a structural material, the physical properties of which must be determined by careful testing; and the results of tests must be intelligently applied in design and construction procedure, in accordance with findings of the science of soil mechanics.

To meet the essential requirements of safety against overtopping, reasonable watertightness, and stability of the foundations and embankment, present-day design of earth dams includes the provision of a much larger spillway capacity than had been customary in earlier years. The embankment has a core or section of low permeability material—with or without a foundation cutoff of soil, concrete, or steel piling—and outer shells of coarser material for the drainage of seepage. The embankment slopes are flat enough to keep stresses sufficiently within the shearing resistance of both the embankment and foundation and to provide an acceptable factor of safety in stability. Obviously, each dam must be "tailor-made," or designed to fit the soils found within a reasonable distance of the site.

Such partial failures as have occurred have generally resulted from overworking local materials in attempts to reach maximum economy. Because earth dams of spectacular height and volume have been built successfully, it does not follow that the design of a low dam is simple. Foundation conditions and the soils locally available for fill may make the design of a dam less than 50 ft high a real problem.

Perhaps the more notable recent trends in design are extreme precaution against overtopping, more extensive preliminary investigation, and recognition of the controlling importance of adequate provision for safe discharge of such seepage as passes through the so-called impervious section.

Both the hydraulic and rolled fill methods of construction continue to be used, but the latter is apparently gaining favor—regardless of the magnitude

of the structure—largely because of the continued development of transportation equipment with larger capacity and higher speed.

Thus, the Garrison Dam—the first of five projected dams on the Missouri River—is 210 ft high with an embankment volume of 75,000,000 cu yd. It is to be built by the rolled fill method, and will be the largest of this type in the world. A spillway capacity of 750,000 cu ft per sec, as compared with a record flood of 268,000 cu ft per sec, is to be provided. The foundation is reported to be sand to a depth of 100 ft, with alluvial clay in some areas, and differential settlement under the heavy loads of the concrete intake structure is recognized as a major problem. The upstream half of the embankment will be impervious rolled fill, and a blanket of the same material will be laid on the valley bottom for a distance of 1,250 ft above the dam. The downstream section of the dam will be of random rolled fill, underlain for drainage by a 10-ft layer of pervious fill extending from the toe to a connection with the impervious section. No cutoff trench in the foundation is proposed, but steel piling will be driven under the upstream slope. The upstream slope is to be 1 on 3 down to a 20-ft berm 80 ft below the top of the dam and 1 on 5 from the berm to the toe. The slope of 1 on 3 is to be faced by 3 ft of stone riprap underlain by 18 in. of gravel and a 10-ft thickness of pervious fill.

The importance of adequate provision for the safe drainage of the shells and of slopes duly related to the shear resistance of the available soil is illustrated by experience at the Saluda Dam in South Carolina. This dam, 208 ft high, was built in 1930 by the semihydraulic fill method. Residual soils were placed on a relatively impervious foundation—with slopes of 1 on 3 upstream and 1 on 2.5 downstream. Shortly after completion, it developed wet spots, sogginess, and sloughing of the downstream slope; and a settlement of 3 ft was observed at one point at the top of the dam. After extensive preliminary investigation, 3,800 ft of 3-ft by 5-ft tunnels, 2,000 ft of 4-ft by 6-ft vertical shafts, and 7,300 ft of ditches were excavated in the saturated sections and filled with stone during the period from 1932 to 1937. In 1943 a contract was let under which a terraced blanket of rock was placed on the downstream slope from a point 40 ft below the crest to the toe, increasing in thickness from approximately 2 ft at its upper limit to more than 30 ft at the toe, and approximating 500,000 cu yd in total volume. Coincidental with this reinforcement of the embankment slope, the capacity of the spillway was increased substantially.

The importance of adequate protection of the upstream slope against wave action—and particularly the importance of a layer of coarse material under paving—is illustrated by experience at the Kingsley Dam on the North Platte River in Nebraska. The 1-on-3 upstream slope of this 140-ft high hydraulic fill dam, built in 1938–1941, was surfaced with pre-cast concrete blocks 4.5 ft long and varying in width from 4.5 in. to 6 in. and in depth from 9 in. to 18 in. These blocks were threaded on two sets of steel rods and placed with a 1-in. space between the ends on an embankment surface, specified to be made up of the coarser material deposited by the hydraulic process but which, as built, contained little coarse material. Before the reservoir was filled, the concrete blocks were undermined by the wave action resulting from a long "fetch," and settlements varying from 2 ft to 5 ft resulted. After an attempted remedy

by grouting and refilling with graded gravel under the moved blocks had been proved inadequate by a later storm, the final method of repair included removal, in greater part, of the concrete blocks, the placing of a 15-in. filter layer of gravel, graded from ¼ in. to 4 in. in size, and a 3-ft outer layer of hard limestone riprap, of which 5% of the individual stones were to weigh 500 lb and 50% 100 lb or more. The original concrete block paving was broken up and used in the riprap. The total outlay for the repair work approximated $2,800,000 (according to another account, $3,165,000) of which one third was for freight charges on filter gravel and riprap stone. The Kingsley Dam experience points to the economic problem of determining how far local materials should be supplemented by materials from more distant sources at greater cost.

The importance of an adequate thickness of finer material beneath paving to avoid washing out of the embankment beneath the paving by the receding waves has been demonstrated by partial failures in other dams but apparently is not too generally known. The finer material itself must be graded so as to prevent its being washed through the joints in the paving. Trouble is generally confined to places where a long fetch produces high waves.

A more liberal use of random stone riprap for drainage and protection of slopes against wave action is indicated by recent practice. Thus, the Davis Dam on the Colorado River, midway between Boulder and Parker dams, is being constructed by the rolled fill method, with a comparatively narrow central impervious section of clayey sand and gravel, intermediate sections of sand and rock screenings, and exterior upstream and downstream sections of rock, 4 in. in size or larger.

The 1-on-2.5 upper section of the upstream slope of the rolled fill Merriman Dam, being constructed on Rondout Creek for the additional supply of New York City, is to be faced with closely placed dry rubble paving 2.5 ft thick, laid on a 5.5-ft layer of random stone riprap; and the 1-on-3.5 lower section of the slope is to be faced with random riprap, increasing in thickness from 10 ft at the top to about 15 ft at the toe. The entire downstream slope of the dam is to be covered by a layer of stone riprap, increasing in thickness from 8 ft at the top to 20 ft at the toe, which is to be surfaced with loam and grass.

Various methods for the control of foundation seepage are being used—steel piling as in the Garrison Dam, an impervious soil cutoff in deep foundation trench as in the Davis Dam, and a foundation trench and concrete caissons sunk to rock as in the Merriman Dam. In general, puddle or soil of low perviousness are preferred to concrete for the control of seepage, and core walls are not generally used unless, as in the Merriman Dam, a deep stratum of pervious material overlies the rock.

A new procedure of some potential value in the control of under-seepage and through-seepage is the excavation and simultaneous sealing of cutoff trenches by clay slurry and backfilling with clay dropped into the unwatered trench. This procedure has been developed by the United States Army Engineers in work on the Mississippi River levees. The keynote of the procedure is the possibility of holding the walls of trenches in pervious material below ground-water level without sheeting by keeping the trench full of clay slurry

during the excavation process. The principle is similar to that used in drilling wells by the rotary method.

Spillways.—The tendency to increase spillway provisions has been noted in several previous reports of the committee and is referred to in the preceding section on earth dams. Provisions for as large a peak flow as $10,000\sqrt{A}$ have been mentioned as outside limits; but, where the reservoir is of considerable area, the effect of pondage is taken into account.

The difficulty in design continues to be lack of flood hydrographs or even unit hydrographs. Recourse is often had to synthetic figures of runoff based on assumed rainfall and the supposed hydraulic characteristics of the streams, but this method is subject to widely varying results according to variations in the judgment of the designer. The U. S. Geological Survey should perhaps be brought closer to design problems and could then make its excellent hydrographic records more responsive to this need.

In the 1940–1942 report,[8] reference was made to the astounding rainfall and runoff records of the Port Allegany storm of July, 1942. The committee has learned of the use of these rainfall records as a basis for the design of a spillway. The results were a spillway capacity of about $8,600\sqrt{A}$.

A general discussion of spillway capacity at the joint session of the Hydraulics and Waterways divisions on January 18, 1945, revealed a tendency of some government experts to advocate provisions for the highest possible flood. The practice of predicting the maximum possible precipitation by use of meteorological data, location, and topography of the catchment area in question seems to have become general for large government projects but is still unavailable for ordinary projects. Most engineers recognize that, for other than huge monumental dams constructed with government money, there is an economic question involved in choice of size of spillway. It may also be stated with truth that the maximum conceivable flood would in most cases be a catastrophe from runoff below the dam even if no breach in the dam should occur; and, if the reservoir is relatively small, the difference would not be very great.

The late Thaddeus Merriman, M. ASCE, once stated that one should make a spillway as large as the Lord permitted. The words were not intended to be taken in their most literal sense—what he meant was that, if nature provided topography favorable to a large spillway, the engineer should take advantage of those conditions.

However, there are considerations other than capacity—namely, those of freeboard versus length. A long spillway crest causes the intensity of all floods downstream to be increased by reducing the effect of pondage back of the dam. Hence, in original design or in improving existing spillways, the proper emphasis would seem to be on high freeboard rather than on excessive length of crest. Improvements to existing spillways which consist merely of increased length of crest might easily be considered by downstream residents as a mixed blessing, to say the least.

The spillway question appears to be one that will never have official or even technical society standardization; the tendency, however, is unmistakably upward in size.

Centrifugal Pumps

A well-known pump engineer has kindly furnished the committee with information quoted or paraphrased in this section. There have been no outstanding changes in the design of centrifugal pumps but refinements in many details have combined to effect considerable improvement in efficiency and in maintenance costs, itemized briefly as follows:

(a) Shaft packing has been given much study, because largely of the requirements of higher pressures. By experiment, mechanical seals have been developed and are being used successfully, particularly on "tough" jobs. The committee's informant forecasts a perfected mechanical seal requiring no attention for months, and giving low maintenance costs and little "outage" for servicing.

(b) Impeller vane angles and internal finish, as well as contours of volute, have been improved.

(c) Higher heads per stage have involved greater stress and improvements in shafts. Carbon steel shafts are still the rule but use of stainless steel of from 11% to 13% chrome type is increasing because of its resistance to wear from grit. Percentage of chrome must not be too high or there is a tendency to seize.

(d) For small-sized and medium-sized bearings, ball bearings are increasingly used with general success. On larger units, sleeve bearings are universally employed.

(e) Couplings continue to be of rubber-bushed type on small units but there has been a change to all-steel, gear-tooth type for medium and large units.

(f) Semisteel is being used increasingly for casings. Government bronze is still widely used for both impellers and case rings.

(g) Investigation of the causes of noise and cavitation has been continued. These are "about the only complaints that can now be found with the operation of centrifugal pumps." Too high speed for the specific conditions is believed to be the cause. "Capacity, head and suction conditions must all be carefully considered if a pump is to be free from cavitation and noise."

(h) "Many operators seem to fail to consider that such improvement has been made in the efficiency in centrifugal pumps in the past 10 or 15 years that pumps purchased 20 to 25 years ago should be replaced because of their higher power costs." The committee's informant believes that pump makers have not emphasized this fact sufficiently.

Valuation of Public Utilities

The 1943–1944 report[10] of the committee contained information on the disposition of the important 1943 and 1944 reports of the Committee on Depreciation of the National Association of Railroad and Utilities Commissioners (NARUC). More complete and accurate information from the NARUC official report of proceedings became available later and will be given briefly.

The 1943 NARUC report recommended and developed in detail the use of straight-line, age-life depreciation based on original cost. In the 1944 report, the NARUC committee reiterated these recommendations but, probably in

[10] *Proceedings*, ASCE, May, 1945, p. 688.

deference to criticisms of those who approved the report in general, sought to explain and soften somewhat the recommendation that deficiencies in depreciation reserves resulting from the substitution of the straight-line method for previously used sinking fund or other methods hitherto unquestioned should be in general recouped by charges against surplus—that is, against any earnings that had not been distributed. If the latter were deficient, any lack would have to be amortized out of future earnings but in general without allowance of correspondingly increased rates.

The NARUC depreciation committee reports of 1943 and 1944 were attacked in reports of a special committee of the Society,[11] by a committee of American Water Works Association (A.W.W.A.), and by various utilities; but the NARUC committee maintained its views.

These 1943 and 1944 NARUC depreciation committee reports were considered at length at the general meeting of NARUC in November, 1944. Two motions were offered:

(1) The original motion, offered by the chairman of the Depreciation Committee, contained the words,

> "Resolved, That the principles and conclusions stated in the said reports * * * be recommended to the Commissions represented in the membership for their consideration and guidance, subject to such modifications in practical application as they, in their respective jurisdictions, may deem necessary in the public interest;" and

(2) A substitute motion worded by the president of NARUC in the interests (as he expressed it) of harmony, contained the wording,

> "Resolved, That the reports of the Committee on Depreciation be distributed to the Commissions represented in the membership of this association for their consideration and application, subject to such modifications in practical application as they, in their respective jurisdictions, may deem necessary in the public interest."

After an interesting discussion, the substitute motion was adopted by the close vote of 19 to 18. (In the foregoing quotations the underlining is by Water Supply Committee.)

The NARUC Committee on Depreciation offered no important suggestions in 1945 or 1946. The chairman under whom the 1943 and 1944 reports on depreciation were developed was appointed to membership in the Federal Power Commission (F.P.C.).

Under the decision of the U. S. Supreme Court in the Hope Natural Gas case, noted in the committee's last report, and another decision in a pipe line case, regulatory bodies feel more free to exercise their own judgment in matters of valuation for rate making and other purposes. It is true that those Supreme Court opinions did not analyze or decide the exact meaning of the inexplicable ambiguity of the 1938 F.P.C. act under which the cases were brought, but they did in effect give the commission a free hand as to the mental process by which it arrived at its conclusions, stipulating only that the results should not be unjust.

[11] *Proceedings*, ASCE, November, 1944, p. 1419.

Subsequent cases that have come before the F.P.C. during 1945 and 1946 are reported to have been settled in the same manner as the Hope Natural Gas case—namely, on a basis of investment cost without reference to "fair value." This tendency may be considered as the present trend in valuation and the results are accentuated by a tendency to give greatest weight to original construction costs and straight-line depreciation. "Fair value" is seldom given weight except in states where the law requires such consideration. This retroactive regulation bears heavily on persons who had previously invested in common and preferred stock when other court decisions were ruling. A recent pronouncement by the F.P.C. seems to confirm the views on present trends as expressed in this report. Referring to an agreement on "cooperative action" with the Arizona State Commission, the F.P.C. states that its rate survey will not include the making of "fair value" but will be undertaken "in accordance with the standard rate-making procedure adhered to by the F.P.C."

Committee of the American Water Works Association on Survival and Retirement

The activities of the A.W.W.A. Committee on Survival and Retirement Experience with Water Supply Facilities should be noted. This committee—among whose members are several (including the chairman) who are also members of the Society—has nearly completed its formal work and has supervised and coordinated the collection of a mass of data on the subject indicated by its title, covering thirteen municipal and eleven private water plants spread over the United States and one municipal plant in Canada. Survival and retirement data on pipe lines, hydrants, valves, services, meters, and other property have been tabulated by years. The results have been published in the *Journal* of A.W.W.A. and will be bound into a volume when paper becomes available.

A considerable part of these data was furnished to the NARUC Committee on Depreciation during its work on its 1943 and 1944 reports.

The conclusion seems inescapable that water supply plant is much more enduring than had generally been assumed previous to this study. It was the contention of some water supply engineers at hearings before the NARUC Committee on Depreciation that there was so little retirement of cast-iron pipe even in the older water works, that there was no basis for computing depreciation by the life-age method. Longer lives are now being assumed generally, based on the data assembled by the A.W.W.A. committee.

Water Purification and Prevention of Corrosion

Chlorine Dioxide.—This process of water treatment is very promising in the field of taste and odor removal, particularly where phenolic compounds are responsible. The equipment is simple but the cost is relatively high.

Deaeration.—Increasing interest is being shown in the deaeration of water for corrosion control. It will be recalled that the first example to be prominently mentioned and described was at a sulfur plant at Grand Ecaille, south of New Orleans, La., where some ten years ago the process was applied very successfully to a steel pipe line.

Plastic Linings.—There are persistent rumors regarding the use of plastics for pipe lining. Small plastic tubes have been used to a limited extent for about six years. At least one type was found to be attractive to rodents, whether for food or as a tooth polish has not been stated.

Demineralization of Water.—Water has been demineralized successfully by ion exchange but this method is largely confined to specialized industrial use. The process is somewhat complicated for public water supply and the cost is too high for any but necessitous cases.

Proportional Chemical Feeds.—There has been a gratifying development in the field of apparatus for the application of chemicals to water, especially in the production of proportioning pumps for solutions of hypochlorite and other chemicals for water treatment.

Cathodic Protection.—The technique for the cathodic protection of tanks has been elaborated in what seems to be a scientific manner and the use of such equipment has expanded rapidly. However, close inspection and accurate comparison of steel surfaces, both protected and unprotected, with all other conditions identical are still lacking. There is room for some real research work as to the effect of cathodic protection with different waters, different degrees of paint protection, different electric current densities, etc. This is not to say that the companies installing such apparatus have not intelligently used the water works tank installations of the United States as a practical laboratory; however, as far as the committee knows, there are no impartial scientific reports of results. The matter of exterior attachments for the application of the cathodic protection is still rather a mystery.

Standards, United States Public Health Service (U.S.P.H.S.).—The new U.S.P.H.S. standards[12] seem to have been well received. Some members of the committee wonder if the child of the future, brought up on heavy chlorine residuals, will regard all chlorine-free water as distasteful.

Respectfully submitted,

Frank A. Barbour	John S. Longwell
John R. Baylis	George D. Norcom
William W. Brush	Thomas H. Wiggin, *Chairman*

Committee of the Sanitary Engineering Division on Water Supply Engineering

January 15, 1947

APPENDIX

CROSS-CONNECTION PROTECTION

By R. F. Goudey,[13] M. ASCE

Before the end of World War II, the armed forces at waterfront areas were making considerable progress in preventing backflow from ship pumps into domestic water systems by installing backflow protective devices at pier-

[12] *Journal*, A.W.W.A., March, 1946, p. 361.

[13] San. Engr., Bureau of Water Works and Supply, Los Angeles, Calif.

head connections. This program has more recently been further promoted by the various state department of health engineers in certifying water for use on interstate carriers. In most instances, the protective device used has been the double check valve.

The University of Southern California at Los Angeles has established a $30,000 foundation to test existing types of backflow protective devices and to set up principles of backflow protection, and proposes to determine specifications for such devices. One of the chief health hazards of any water distribution system is the possible pollution through cross connections at times when backflow conditions have become established. It is of little avail to outlaw the existence of cross connections if the incentives to take advantage of the quantity or pressure of the domestic supply have not been removed. Water piping systems must terminate in fixtures or water-using devices which in their inherent nature are cross connections that cannot be eliminated. It is necessary, therefore, to rely on mechanical devices which should be designed for different conditions in proportion to the danger or seriousness involved.

The most promising general type of backflow device for use under pressure is the equivalent of a double check valve installation such that the pressure in the intermediate zone is always maintained below that of the incoming supply. This is equivalent to "air gap" protection. Such devices, when standardized, provide full protection up to the degree of inspection service rendered, which is considered safer than trying to rely on absolute physical separation, which becomes zero whenever an unprotected cross connection is made.

In the Los Angeles Harbor area, more than $3,500,000 has been spent since 1942 in protecting all waterfront piping systems against backflow. In addition, protective devices have been installed on all fire service and domestic services at the property line.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE MARINE OPERATING PROBLEMS, PANAMA CANAL, AND THE SOLUTION

Discussion

BY THE EDITOR

Since the printing of this paper in February, 1947, *Proceedings*, a plan has been approved by the Society's Committee on Publications for enlarging and simplifying the entire treatment of this subject. One immediate result will be to defer the printing of discussions. Their submission is meanwhile encouraged, particularly while the interest in these subjects is high, even though the actual appearance of the discussions in print may be delayed.

The paper has been "in the works" for many months. Meanwhile a Congressional directive has started an intensive official study of the entire broad Panama Canal development. An eminent engineering consulting board, as reported in October, 1946, *Civil Engineering* (page 471), in addition to the regular Panama Canal staff, has been busy on this work under direction of the Governor General J. C. Mehaffey. Out of this study are expected to come a number of technical papers of special interest to civil engineers. These, however, cannot be released until the report itself is submitted to Congress toward the end of the year.

Therefore, the Society is cooperating in arrangements looking toward a broad symposium on the Panama problem at its January, 1948, meeting, covering plans additional to the present Society papers. Because the paper by Captain DuVal is thus restricted for a short time, the Committee on Publications has concluded that discussion of the over-all subject would be more enlightening and productive if all printed discussion were deferred until such time as it can encompass the expected as well as the presently available papers.

If for any reason the engineering report of Governor Mehaffey to Congress is delayed beyond December 31, 1947, it is the intent of the Committee on Publications to release forthwith any waiting material on the Panama problem, including all discussions that may be available. It is hoped, however, that by holding over the intervening discussions, printed comments on the proposed symposium as well as on the DuVal paper may come out concurrently after January, 1948.

NOTE.—This paper by Miles P. DuVal was published in February, 1947, *Proceedings*.

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DISCUSSIONS

SEA LEVEL PLAN FOR PANAMA CANAL

TO PROVIDE MAXIMUM SAFETY AND UNLIMITED CAPACITY

Discussion

BY THE EDITOR

Discussion on this paper is invited. By action of the Society's Committee on Publications, the actual printing of such discussion is to be deferred and presented concurrently with comments expected later but now withheld. The arrangement is the same as that for the paper by Miles P. DuVal, Esq., as mentioned in more detail on p. 523.

NOTE.—This paper by J. G. Claybourn was published in February, 1947, *Proceedings*.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

FUTURE COSTS AND THEIR EFFECTS ON ENGINEERING BUDGETS

Discussion

BY LOUIS R. HOWSON

LOUIS R. HOWSON,[29] M. ASCE.[29a]—The author is in agreement with the first premise of Mr. Mendelsohn's discussion, namely, that construction costs are rising. He cannot agree with the second statement that they are in a "stage of unpredictable flux." All engineering construction costs are necessarily a matter of prediction. Nothing can be built in the past and therefore the engineer is directly concerned with future construction costs. The past serves only as a basis for his estimate of the costs and conditions under which work must be executed in the future.

That it is practicable to predict with reasonable accuracy the effect of wars upon construction costs is well indicated by the forecast made by the late John W. Alvord, Hon. M. ASCE, reproduced as Fig. 3. In this forecast, made in 1920, Mr. Alvord predicted not only the lowest level which prices would reach following World War I, but also predicted that the low point would be reached 12 years after the date of his forecast, or in 1932.

Mr. Mendelsohn indicates that the author's analysis of factors of construction cost is inadequate in that the relationship of wage rates to high construction costs is believed unduly stressed. Reply to this suggestion is offered by reference to Fig. 5, which shows the relation between wage rates and construction cost from 1913 through 1945. The fact that Fig. 5 discloses the generally close relationship cannot be disputed either during a period of rising or a period of declining costs. For illustration, in the 33-yr period from 1913 to January, 1946, the *Engineering News-Record* (E.N.-R.) construction cost index increased from 100 to 316, while the average hourly wage rate of skilled and common construction labor increased from 100 to 353 in the same period.

In the shorter upswing during World War II—that is, from 1939 to April, 1946—the E.N.-R. construction cost index increased 42%, whereas the average of common and skilled labor increased $34\frac{1}{2}$%. During 1946, as reported

NOTE.—This paper by Louis R. Howson was published in March, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1947, by Isador W. Mendelsohn.

[29] Cons. Engr. (Alvord, Burdick and Howson), Chicago, Ill.

[29a] Received March 3, 1947.

in *Engineering News-Record* for February 20, 1947, the E.N.-R.construction cost index rose 12.3% while skilled construction labor wages rose 10.8% and common labor wages rose 15.7%.

Similarly, at the beginning of the depression from 1930 to 1932, the E.N.-R. construction cost index decreased 23.1% while during the same period the average reduction in common and skilled labor rates was 25.2%. On the long upswing from 1932 to April, 1946, the E.N.-R. construction cost index rose 114.5% as compared to the average increase in common and skilled labor hourly wage rates of 103.5%.

In this connection it was stated in the paper under the heading, "4. Effect of Wars on Price Levels:"

> "It should be noted that Fig. 5 [relation between wage rates and construction cost] does not reflect the result of post World War II strikes, wage increases, and cost increases. These might well raise graph B [E.N.-R. construction cost index] to 350 or 360 as compared to its 1945 figure of 310 * * *."

It is significant that as of January 1, 1947, one year after the paper was presented to the Society, the E.N.-R. construction cost index was 381. This short term prediction of the effect of strikes, wage increases, and cost increases also indicates that construction costs are not wholly in a stage of "unpredictable flux."

Mr. Mendelsohn quotes from Beardsley Ruml, Thurman Arnold, Harold L. Ickes, and others not experienced in engineering construction to develop what is apparently his premise that there are vested interests among contractors and materials' suppliers operating in restraint of trade and that the result is higher costs in the building industry than would otherwise be necessary. The author would call attention to the fact that not the least of the "vested interests" is organized labor under the control of some labor leaders.

Since Mr. Mendelsohn has based much of his discussion upon statements of those not in the construction industry, the author presumes to reply in kind from an editorial from the *Chicago Daily News* entitled "No Building." The following is quoted:

> "In the face of these demands, Northwestern University [Evanston, Ill.] has been forced to abandon a $3,000,000 building program—including dormitories, classrooms and hospital facilites—because building costs have increased 75% during the last year. Loyola University [Chicago, Ill.] is considering a two year moratorium on building. The University of Illinois [Urbana] is in process of curtailing a huge building program to the irreducible minimum.
>
> "There is another example of how high material prices and high wages in the building trades, and monopolistic control by union labor works against the public interest. The 75% increase in building costs is due in part to the increase in building wages; in part to the increased cost of materials. The increased cost of materials, in turn, is largely due to the increased cost of the labor that goes into their production.
>
> "Because the building trades unions have a tight monopoly over all building operations, new materials and new construction methods that could save time and money are banned. Because organized labor has a monopoly, it is able to keep the number of building tradesmen far below the demand, thus maintaining an artificial scarcity of labor.

"Because of those conditions, construction necessary to the public welfare cannot go ahead. Monopolies in restraint of trade are forbidden by law; union labor operates in restraint of essential building. But union labor is exempt from regulation as a monopoly."

Mr. Mendelsohn calls attention to the effect of building codes upon construction costs. Certainly there is much room for improvement through the modernization of building codes.

Mr. Mendelsohn discusses the E.N.-R. construction cost index at considerable length. This index obviously does not attempt to reflect the prices resulting from the iniquitous black market practices in the sale of construction materials which are referred to by Mr. Mendelsohn as existing during the period of the late Office of Price Administration.

From Mr. Mendelsohn's statement with respect to the dependability of the E.N.-R. construction cost index in which he refers to a paper by the author reported elsewhere[26] it might be inferred that the author does not consider the E.N.-R. index reliable. That is far from the fact. To clarify the author's position the entire paragraph referred to by Mr. Mendelsohn is reproduced as follows:

"Construction Costs vs. Construction Cost Indexes

"In normal times, the *Engineering News-Record* Construction Cost Index is a valuable and reliable aid in translating costs of one period into estimates for another. However, 1946 is not a normal year. The construction industry is beset by fear—fear of strikes, uncertainty about labor rates and efficiency, inability to secure materials when needed and to contract for materials at a fixed price before bidding, and fear of government- and union-imposed restrictions affecting performance and costs. These all operate to restrict bidding and increase the allowance for contingencies and the total cost. These hazards are not reflected in construction cost indexes. Accordingly, in 1946 it is practically the universal experience of those attempting to let work by contract that the prices bid much exceed those estimated by adjusting prior construction costs according to the construction cost indexes."

Comparison of labor performance in building construction and in mass production industries cannot be fairly made. Buildings are, and to a major extent always will be, individual units built at scattered locations and each is essentially "custom built." There is no production line in general building construction. Production line methods do not apply to building construction and, particularly with respect to dwellings, standardization appears not to be desired by those who need living accommodations.

The author agrees with Mr. Mendelsohn that good planning results in lower costs but good planning is essential at all times and cannot balance increased labor wage rates and material costs. In studying construction cost trends it must be presumed that every engineering project has had the benefit of economic study and development.

The author appreciates the analysis of construction cost factors and their development in Mr. Mendelsohn's discussion.

[26] "Construction Costs and Water Rates," by Louis R. Howson, *Journal*, A.W.W.A., June, 1946, p. 747.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

MOMENT-STIFFNESS RELATIONS IN CONTINUOUS FRAMES WITH PRISMATIC MEMBERS

Discussion

BY WILLIAM A. CONWELL

WILLIAM A. CONWELL,[5] M. ASCE.[5a]—The merit of this paper lies principally in the fact that it extends the theoretical horizon of moment distribution. Many engineers, faced with the prospect of repeated analyses of a problem by moment distribution, have cast about for a more direct method of obtaining results. Their search usually brought them into contact with a variety of equations which separated the designer's mind from the physical characteristics of the structure under consideration. Since "avoidance of such general equations was the object"[6] of moment distribution, the hunt usually ceased at the first encounter with algebra, and a quick retreat to moment-distribution fundamentals was effected. The author, however, was rewarded when he continued his probing after first discouragements. Although use of the moment-stiffness relations does involve formulas, these latter are intimately associated with the fundamental physical characteristic of the change in bending moments caused by a change in stiffness and, if only from this point of view, are worthy of study.

An inspection of Table 1 is enlightening in this regard. It will be noted, for instance, that a stiffness-change ratio, i_2, of 10% in member AB produces a change of -8.44 in M_{BA}; but an equal 10% stiffness-change ratio, i_4, in member BC produces a greater change, $+11.08$, in M_{BA}. Similarly, a 10% value of i_1 in member AA′ produces a change of $+7.10$ in $M_{AA'}$, which is less than the -12.38 change produced by a similar 10% value of i_2 in member AB. It is but another step to observe that the effect of a change in stiffness on a bending moment does not necessarily increase with the proximity of the member in which the stiffness change occurs. For example: A 10% value of i_1 in member

NOTE.—This paper by George H. Dell was published in May, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1946, by I. Oesterblom.

[5] Gen. Engr., Structural Eng. and Design Dept., Duquesne Light Co., Pittsburgh, Pa.

[5a] Received January 31, 1947.

[6] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, p. 141.

AA′, the one most distant from member CC′, produces a change of +1.84 in $M_{CC'}$, which is almost as great as the +1.94 produced by a 10% value of i_4 in the adjacent member, BC. From the foregoing it may be readily concluded than a change in stiffness in one member cannot be made without the possibility of a profound effect on even the most distant members of the frame.

With a view to comparing a moment-stiffness analysis with a direct analysis, as regards amount of labor, rapidity of convergence, and other items, the writer designed the rigid frame of Examples 3 and 5, employing direct methods only. The third direct analysis—that is, the second after the original—produced the values given by the author in step (7), Example 5, as the results of his final direct analysis used to check the moment-stiffness method. The only variance of any significance in those data was in the 132.0 kips obtained for F in member AA′ as against 133.7 kips. Accordingly, in Table 4, the author's

TABLE 4.—COMPARISON OF RESULTS OF MOMENT-STIFFNESS AND DIRECT ANALYSES IN EXAMPLE 5

Type of Load	Member	First Cycle		Second Cycle				Third Cycle		
				Moment Stiffness		Direct		Moment Stiffness		Direct
		Value	Error, %	Value	Error, %	Value	Error, %	Value	Error, %	Value
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Axial	AA′	136.5	+2.1	136.5	+2.1	132	−1.3	136.5	+2.1	133.7
	BB′	297.8	−1.6	297.8	−1.6	303	−0.1	297.8	−1.6	302.7
	CC′	92.8	−0.9	92.8	−0.9	94	+0.4	92.8	−0.9	93.6
End moment	AB	822	−5.2	867.1	0	865	−0.2	869.3	+0.3	867
	BC	755	−5.1	787.1	−1.1	798	+0.3	786.2	−1.2	796
	AA′	238	+29.3	188.6	+2.5	192	+4.3	186.8	+1.5	184
	BB′	133	−24.4	165.0	−6.2	166	−5.7	179.4	+1.9	176
	CC′	132	+16.8	120.4	+6.5	120	+6.2	115.9	+2.6	113

values were used as the final correct figures, and the several stages of the two methods of analysis are compared with the final results. As may be seen, each successive cycle of the direct analysis steadily approaches the final figures. The second cycle of the moment-stiffness analysis compares favorably with that of the direct analysis; but the third cycle of the former is erratic, when compared to the complete conformance of the latter with the final results. The differences are, of course, small in magnitude, insufficient to produce changes in section, and easily explainable by the unusually great differences between the stiffness of certain members in the final design and those in the basic structure.

As to the comparison of the amount of work, the direct method required four analyses for each cycle—one each for uniform loads on members AA′, AB, and BC, and one for a horizontal load at the top of the frame—a total of twelve analyses. The moment-stiffness method initially requires four analyses, one each for moments m_A, m_B, m_C, and m_D. If the recommendation in the "Synopsis," that "the final results should generally be checked by direct analysis" is followed, four additional analyses will be required, making a total of

eight. Between the initial and final analyses, the moment-stiffness method requires two series of substitutions in Eq. 7 for changes in the controlling moments in each member. A comparison of the methods in this instance would seem to indicate that the choice is between four direct analyses of the structure and ten substitutions in Eq. 7. Both methods require a certain amount of superposition and proportioning of analyses, but these would largely balance on the work ledger sheet. The decision as to method would probably be one of personal preference, since the labor involved appears to be about the same. Elimination of one condition of loading would throw the balance sharply in favor of the direct method as one less analysis would be required in each cycle, whereas the number of analyses for moment stiffness would remain the same. On the other hand, an additional condition of loading would not be as favorable to the moment-stiffness method since, in the direct method, use can be made of m_A, m_B, m_C, and m_D computed for each cycle and the number of analyses per cycle can be held to a maximum of four.

The second paragraph under step (4), Example 5, should be carefully followed in conjunction with Table 2 to be readily understood. However, the appreciation of designers is gained by calling attention to the check that the sum of coefficients of i in any expression for dM equals zero. Another valuable feature is the analysis of multiple-span peaked bents in "Appendix 2."

Finally, the author has exhibited a fine sense of balance in presenting the paper for what it is worth, allowing it to stand on its merits and making no claims that cannot be substantiated by a reading of the paper and practice in use of the methods.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

CRITICAL STRESSES IN A CIRCULAR RING

Discussion

BY E. A. RIPPERGER AND N. DAVIDS

E. A. RIPPERGER,[26] ESQ., AND N. DAVIDS,[27] ESQ.[27a]—Messrs. Philippe and Mellinger have described one practical application involving the use of the theoretical calculations contained in this paper, and, as Mr. Popov has suggested, there are no doubt many others. The wide variations in the nature of the stress distribution at the so-called critical section, as illustrated in Fig. 2, make it necessary, however, to use a great deal of caution in applying these theoretical results to the calculation of failure stresses for even relatively brittle materials. To illustrate and emphasize this point the data in Table 6 have been compiled. These computed failure stresses for concrete rings having the dimensions shown were calculated through the use of Eq. 11.

TABLE 6.—COMPUTED FAILURE STRESSES FOR CONCRETE RINGS[a]

DIMENSIONS, IN.			FAILURE STRESS, POUNDS PER SQUARE INCH				
r_i (1)	r_o (2)	t[b] (3)	Specimen 1 (4)	Specimen 2 (5)	Specimen 3 (6)	Specimen 4 (7)	Average (8)
2	6	4	1,230	1,390	1,290	1,240	1,287
1	5	4	1,780	1,790	1,750	1,820	1,785
½	3	1	2,480	2,390	2,610	2,160	2,410
Compressive strength[c]			6,720	6,680	6,630	7,290	6,830

[a] Unpublished results of an investigation by I. Narrow, Ohio River Division Laboratories, Mariemont, Ohio.
[b] Thickness of ring.
[c] Compressive strength, in pounds per square inch, of 6-in. by 12-in. cylinder.

Specimens were moist cured for 28 days before testing and the maximum size of the aggregate was approximately ¾ in. The apparent increase in tensile strength as the value of r decreases is believed to be due chiefly to the decrease

NOTE.—This paper by E. A. Ripperger and N. Davids was published in February, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1946, by Robert H. Philippe and Frank M. Mellinger; and November, 1946, by E. P. Popov.

[26] Dept. of Eng. Mechanics, Univ. of Texas, Austin, Tex.

[27] Math. Dept., Johns Hopkins Univ., Baltimore, Md.

[27a] Received February 13, 1947.

in linearity of the stress distribution at the critical section as $\bar{r}$ becomes small. A small plastic yielding of the concrete at the critical point enables the ring with $\bar{r} = 0.1$ to withstand a much larger increase in the load (P) required to cause failure, than a similar deformation would cause in a ring with, say, $\bar{r} = 0.3$. It will also be noted that variations of calculated strength from specimen to specimen are greater for $\bar{r} = 0.1$ than they are for larger values of $\bar{r}$. This may be attributed to the fact that, for the case where $\bar{r} = 0.1$, the hole in the ring represents a discontinuity of the same order of magnitude as the discontinuities introduced by the larger aggregate particles.

Mr. Popov implies that the use of coefficients taken from the nonconvergent boundary stress series in the development of a convergent series to represent interior stresses is a questionable procedure which should have been avoided by the use of Professor Timoshenko's method. Although it is certainly true that nonconvergent series must be used with much caution in mathematical processes, the procedure followed in this paper is shown to be quite correct (see the discussion of that point and the continuity proof, Eqs. 14, 15, 16, and 17) and at least a part of any value this paper may have lies in that demonstration. Despite the general mistrust of nonconvergent series by mathematicians, several problems in mathematical physics have been correctly solved by such series.[28] The ring problem is another example.

The question as to whether or not a loaded circular ring actually represents a case of plane stress does not arise in theoretical treatment of the problem, for it is assumed that the load is distributed in such a way as to give a plane-stress condition. The question does arise, however, when any practical application of the calculated results is contemplated, and Mr. Popov is to be commended for having posed the question and then answered it.

The authors are indebted to Mr. Popov for having brought to their attention some excellent work done on this problem, of which they were unaware at the time of their own investigation.

Corrections for *Transactions:* In February, 1946, *Proceedings*, on page 161, line 8, change "and, for $\theta \neq 0$ and $\theta = \pi$," to "and, for $\theta \neq 0$ and $\theta \neq \pi$"; on page 161, line 15, change "* * * after substituting $\theta = 2\,\pi$," to "* * * after substituting $\theta = 2\,\phi$"; on page 163, Table 1, change the entry for $\bar{r} = 0.5$, $r/r_o = 0.8$ from 1.6168 to 1.1668; on page 166, Eq. 19 should read

$$\underset{r_i \to 0}{\text{Limit}}\ \sigma_\theta \Big]_{r=r_i} = \frac{6\,P}{\pi\,r_o};$$

and on page 166, Eq. 20 should read $\sigma_\theta = \dfrac{P}{\pi\,r_o}$.

[28] "An Introduction to the Theory of Infinite Series," by T. J. I. Bromwich, Macmillan Co., London, p. 317.

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DISCUSSIONS

EXPRESS HIGHWAY PLANNING IN METROPOLITAN AREAS

Discussion

BY M. HIRSCHTHAL, AND JOSEPH BARNETT

M. HIRSCHTHAL,[29] M. ASCE.[29a]—The profession is greatly indebted to Mr. Barnett for his timely and substantial contribution, which focuses attention on a great need—a thorough discussion of means of coordinating express highway traffic through cities and suburbs of a metropolis so as to produce a free flow throughout.

Development of the highway and its design have naturally followed closely upon the development of the automobile. When the automobile first made its appearance, a 16-ft width of roadway constituted a two-lane highway. This was shortly increased to 18 ft with an allowance of 9 ft for a width of lane; a four-lane highway was made 36 ft wide. At that time the general practice was to use 60% of the street width or right of way as roadway, thus providing a 36-ft road width for a 60-ft city street or highway. Most city streets have been laid out on this ratio basis, even to the 100-ft-wide express streets in which the roadway has a width of 60 ft. Lane widths in state highway design specifications were set at 9 ft as standard, and remained so until about 1941.

In recent years this lane width has been gradually increased to 12 ft (particularly on express highways), with a provision of 5 ft or 6 ft (in some instances, as much as 12 ft) for a "parkway" separation between the two sets of lanes for traffic in opposite directions, and with additional provisions for gutters and shoulders on either side. (Incidentally, in all this express highway planning, including New York and New Jersey state highways, nowhere has the writer seen any provision for the pedestrian—he is the forgotten man.)

Despite the widening of the lanes, perhaps because of such widening, it is not at all unusual to find cars straddling two lanes. Accidents continue to increase by reason of cars moving from one lane to another, or by passing an-

NOTE.—This paper by Joseph Barnett was published in March, 1946, *Proceedings*. Discussion on this paper has appeared as follows: May, 1946, by Harry W. Lochner, and Fred Lavis; September, 1946, by Homer M. Hadley, Donald M. Baker, W. J. Van London, Merrill D. Knight, Jr., and R. H. Baldock; November, 1946, by Jacob Feld, Harold M. Lewis, Theodore T. McCrosky, Spencer A. Snook, Lawrence S. Waterbury, Bernard L. Weiner, and George H. Herrold, and January, 1947, by Ralph R. Leffler.

[29] Concrete Engr., D. L. & W. R. R., Hoboken, N. J.

[29a] Received January 24, 1947.

other car and miscalculating the space available for that purpose, especially if a bus or a truck is one of the vehicles involved. The only way to avoid these all too frequent occurrences is to provide barriers between the lanes of traffic in the same direction; to prevent such crossovers except at definite locations, say, at one-half mile (or perhaps one-quarter mile) intervals, with proper safeguards similar to those for the many points of merging traffic from roads entering into the express highway. The writer anticipates electronic control in the future.

Such barriers could be provided, without the necessity of additional right of way, by making 9 ft the clear lane width, the actual requirement for a vehicle. The remaining 3 ft of the 12-ft lane widths could be used for barriers or curbs of sufficient height (15 in. minimum) to eliminate the possibility of vehicles mounting these curbs and at the same time to discourage the motorist from driving too close to them. Furthermore, this would simplify the assignment of a definite lane for trucks and other slow moving vehicles if desired. Hazards of sideswiping and similar accidents would be reduced to a minimum. Another advantage is attained in the economy of the cost of grade eliminations across the highway, both for railroads and other highway overcrossings and under-crossings, which would more than compensate for the extra cost of the barriers and the provision of drainage.

Provision of the 3-ft-wide barrier between the traffic lanes in the same direction permits locating in that space the columns or piers required for a bridge in the event of grade crossing elimination over the express highway (a railroad overcrossing) and eliminates the necessity of long-span construction, which is evidently far more costly than that for short spans. Also, where soil conditions are poor, it may obviate the necessity of expensive foundations by the substitution of a spread footing. This may be all that is required to carry the light reactions resulting from short-span design, while heavy foundations are likely to be required for piers and abutments that carry the loads from long spans. Moreover, the great majority of crossings between highways and railroads are at angles other than right angles, resulting in skew bridge structures of increased span length, thus accentuating the foregoing condition.

To cite an actual example: At Fox Hill, Mountain Lakes, N. J., state highway route No. 6 crosses the tracks of the Delaware, Lackawanna and Western Railroad (D.L. and W.R.R.) at an angle of 26° 19′ (a skew of 63° 41′), requiring a center-to-center span length of girder of 103 ft to span the required lanes in one direction; whereas each lane would require only one quarter of such span length had it been possible to provide these barriers and hence space for the location of columns or piers between each lane. It is quite evident that 103 lin ft of girders for four 26-ft spans would weigh a fraction of that for one 103-ft span, particularly for railroad loading, and that the sum of the reactions of the small spans would be similarly reduced for the footing design.

Where the express highway is the overcrossing in a grade elimination, the provision of this space for barriers will make available shallower floor depths because the floor system need span only the width of one lane with girders located in the barrier space, and the 15-in. or more available depth above the road surface may prove sufficient for a depth of girder of moderate span length, so that the girder itself (or a pair of girders) may act as the required barriers.

Similarly there can be cited an example of this case: New Jersey State Highway route No. S3 crosses the D.L. and W.R.R. tracks at Clifton, N. J., on a proposed bridge over the four railroad tracks. The plans provide for the girders to be 54 ft apart on each side of the 16-ft center "parkway" or "island" to accommodate the lanes in each direction. For this condition a $4\frac{1}{2}$-ft floor depth is required. It is evident that a drastic reduction in this floor depth would result if the girders spanning the tracks were only 12 ft apart (in the barrier space) with additional resulting economy in the length of approaches required for this reduced floor depth. Numerous cases could be cited, but these have been selected because of their emphasis.

Approximately 25 years ago the writer proposed[30] to the New York City (N. Y.) authorities a remedy for the traffic congestion on the avenues running north and south, in the form of elevated sidewalks providing two additional traffic lanes, instead of the 25-ft sidewalk space, in each direction for local traffic, with pre-cast bridges spanning the intersecting street roadway openings. It was then emphasized that the shops would benefit thereby in having two stories of display to advertise their merchandise. A a matter of fact, this remedy could be applied as well to all the crosstown streets now so congested in midtown Manhattan.

The writer hopes these comments will receive the attention of express highway planners in metropolitan as well as other areas.

JOSEPH BARNETT,[31] M. ASCE.[31a]—Although the sixteen discussions are chiefly confirmations of the principles of express highway planning as given in the paper, some are critical. Mr. Lavis suggests that the profession would derive more benefit if the paper were more specific. It is agreed that each city and each project in a city needs detailed study; yet the solutions of specific problems show a definite sameness that leads to a set of principles. This is time-honored engineering procedure. The paper is intended to show these principles.

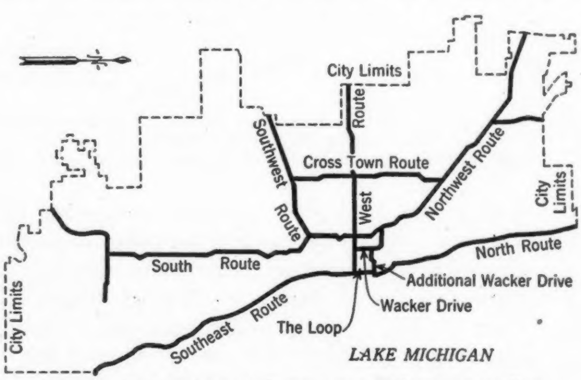

FIG. 13.—COMPREHENSIVE SUPERHIGHWAY SYSTEM FOR CITY OF CHICAGO, ILL.

As a matter of fact Figs. 1, 2, and 3, showing patterns of arterials, are actual cases although the names of the cities are not given. Figs. 4, 5, and 6 illustrate types of expressways which have been in operation for some years.

[30] "Elevated Sidewalks of Reinforced Concrete for New York City," by M. Hirschthal, *Engineering News-Record*, May 31, 1923, p. 974.

[31] Chf., Urban Road Div., Public Roads Administration, Federal Works Agency, Washington, D. C.

[31a] Received February 17, 1947.

An excellent example of the pattern of arterials described in the paper (namely, radial routes and an inner circumferential, with an intermediate or outer circumferential for the larger cities) is presented in Fig. 13, which shows the arterials for Chicago, Ill. The locations were determined over a period of years after factual origin-destination data were obtained and analyzed and the other factors outlined in the paper had been given serious consideration. A few of the arterials, such as the Lake Shore Drive, are in operation and the West Expressway is under construction.

The fact that this pattern is ideal in most respects is confirmed by the discussion of Messrs. Baker and Van London, who cite specific cases in which a large percentage of the traffic traversing the central business district is destined beyond it. An analysis of seven cities from this standpoint is shown in Fig. 14. By carrying the large portion of traffic destined beyond the central business district on free-flowing arteries, congestion of the downtown streets will be relieved and traffic necessarily using them will be better accommodated. On this factual basis Mr. Herrold's criticism is invalid. When paralleled by solutions to other problems, such as the terminal problem, there can be little doubt that relief of downtown traffic congestion will result. These facts are overwhelming arguments against the policy of carrying free-flowing arteries to the edges of central business districts and expecting existing streets to act both as distributors and as arteries for through traffic. These streets generally already are used to capacity and superimposing upon them the traffic induced by improving radial routes will further congest the downtown streets.

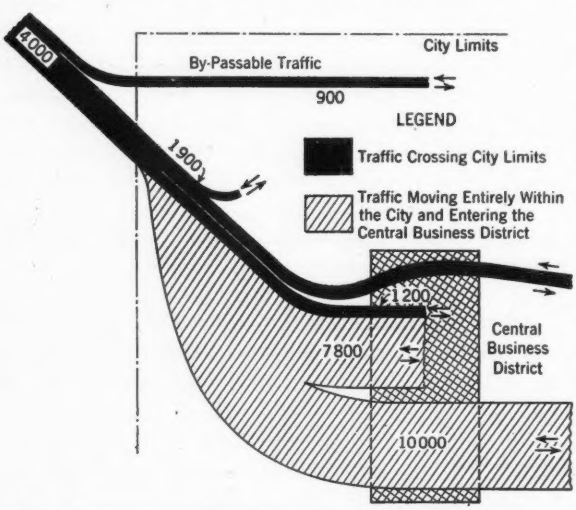

FIG. 14.—ORIGINS AND DESTINATIONS OF TRAFFIC ON AN INTERSTATE ROUTE THROUGH A TYPICAL CITY WITH A POPULATION OF FROM 50,000 OT 100,000, AS BASED ON DATA FROM SEVEN CITIES

A careful examination of the effects of a proposed solution on the downtown streets was discussed by Messrs. Baldock and Lewis and more attention to the terminal problem was suggested by Messrs. Baldock and Weiner. Both problems are related. No analysis of a city motor transport problem is complete without an analysis of the effects of heavily traveled free-flowing arteries on existing streets, an analysis of the terminal problem, and what should and can be done about both. Without such analyses a city may well find that, as Mr. Herrold stated, although the arteries will enable drivers to reach the central business district sooner, greater congestion may well occur. However, the

opposite will result if a properly coordinated plan, based on factual data and not just opinions, is carried through. In Atlanta, Ga., for example, the parking desires and the prospective parking facilities had a profound effect on the choice to locate an important arterial on one, rather than on the opposite, edge of the central business district. The latter location would have resulted in vehicles destined for the parking areas using streets through the central business district whereas with the former location vehicles destined for parking areas will travel short distances on existing streets not used as much as the downtown streets.

Proper locations of arterials, traffic and parking desires, and effects of proposed solutions on existing streets cannot be safely determined by measurements of volumes on existing thoroughfares and the judgment of planners, even when backed by the customary data collected by city planners. These are valuable assets but they cannot be relied upon without factual data regarding origin and destination, parking and loading, and the many important needs revealed by the obtainment and analysis of such data. Expenditures for improvements of arterial routes in cities are large. Not only are initial expenditures high, but a location once accepted and agreed upon commits public officials to a long-range program of improvement involving comparatively large sums of money. The foundation for such a program should be sound justification based upon factual data; benefits to traffic, both arterial and other traffic affected thereby; and benefits to the city as a whole.

Factual origin-destination and parking data are not difficult to obtain and analyze and are not costly in the light of the improvement expenditures involved. The comprehensive home interview type of origin-destination survey has been undertaken in nearly sixty cities and the "bugs" have nearly all been eliminated. The cost is about 10¢ per person in large cities of 1,000,000 population or more. Cost per person increases as population decreases, reaching about 17¢ for the cities in the population group of from 50,000 to 100,000. A parking survey in the downtown area, generally needed to supplement the origin-destination survey, costs about $20,000 additional when made in cities in the larger group.

In not all cities are the cost, time, and effort of a comprehensive home interview type of survey justified. Although this method has been used successfully in cities with a population as small as 15,000, reliable origin-destination data adequate to establish the locations of arterial routes and to indicate the general type of design can be obtained in small cities at less cost by other methods. By one method, drivers of vehicles are interviewed as they are parked. Data are thus obtained not only as to parking habits and desires but also as to the origin and destination of the trip and the purpose thereof. Such surveys cost from $4,000 to $8,000. In another method origin-destination data are obtained from vehicles passing two or more cordons of survey stations. One cordon generally is placed around the edges of the urban area and another, around the central business district. The data are best procured by interviews, although postcard questionnaires are sometimes used. This method is successful if knowledge regarding trips between different sections

of the city outside the central business district does not have an important bearing on the problem.

The value of adequate and reasonably accurate traffic data lies beyond the determination of desirable locations of arterial expressways. In many cases locations of arterials can be determined with very meager traffic data; but an analysis of the effects upon the existing street system, previously discussed, and a determination of the accesses and the traffic loads for which they should be designed cannot be made properly without such data.

The matter of terminals, both for passenger vehicles and trucks, is a vast subject and the statement of principles in the paper is condensed indeed. The problem has been excellently restated by Nathan Cherniack, M.ASCE, and by discussers of his paper.[32]

In regard to design types, Messrs. Hadley and Leffler bring up the matter of elevated structures, a two-level structure being one type suggested. It may be of interest that the elevated highway shown in Fig. 6 (which was designed many years ago) provides a second level, but this idea has been abandoned. The two-level elevated has the advantage of narrowness and economy as compared with a one-level elevated, but the ramps leading to the upper level present added problems and damage to adjacent land may be great. The advantages and disadvantages of elevateds as given in the paper are valid regardless of the number of levels. Several designs that began by considering the elevated type as obviously superior finally were designed as depressed expressways. This was not because the engineers were "pavement minded" as accused by Mr. Leffler. If they were, the elevated would not have been thought of initially. Very few elevated highways have been constructed; on these, wide right of way was available or was acquired.

Messrs. McCrosky, Hadley, and Waterbury suggest the possibility of direct connections between expressways, particularly elevateds, and parking garages. This possibility has been discussed widely because it appears to have merit. Traffic using such garages would not be delayed at terminals and would not use existing streets, thus reducing the volume of traffic thereon. However, a word of caution is in order. Assuming that all the practical and financial difficulties of such a development can be overcome, consideration must be given to the effect of operation on expressway traffic. The fundamental concept of an expressway is free movement and infrequent points of access. The rate at which vehicles can be moved into parking areas or garages is not great at best, so that numerous such points of ingress and egress would have to be provided unless an extremely large storage plaza is available. The plan does not appear to have the flexibility of operation that is desirable to insure against interference with traffic on the expressway. A more flexible plan, and one which appears readily attainable, is one in which the parking facilities are near the expressway but are not a part of it.

Mr. Lewis and Mr. McCrosky discuss the desirability of expressways for segregated traffic. It is agreed that there is fallacy in the theory that the construction of expressways or parkways for the exclusive use of passenger vehi-

[32] "A Statement of the Parking Problem," by Nathan Cherniack, *Proceedings*, Highway Research Board, National Research Council, Vol. 25, 1945, p. 250.

cles will benefit commercial vehicles sufficiently by releasing the existing streets to them; yet the segregation of traffic is desirable because of the smooth operation, high capacity, and great benefits to passenger vehicles and occupants including passengers in buses (where use by buses is feasible). In the larger cities separate expressways for passenger vehicles and for mixed traffic appear to be justified.

Several discussers take up the matter of transit. Improvement of transit is one of the important items in the broad attack necessary to relieve congestion in the cities. It is unrealistic to expect most of the home-to-work, home-to-shop, and home-to-business trips to be made in private vehicles, particularly in the larger cities. When properly designed, operated, and controlled, mass transportation can do much to relieve traffic congestion simply by reducing the number of vehicles on the streets and in the parking places. However, the position of express highways in relation to transit is not as simple as the discussers indicate.

In the larger cities rail transit might be included in the same right of way with expressways with some economy resulting to both—but such a case is certain to be exceptional. Use of expressways by buses is the more likely pattern into which transit will fit, particularly for rush hour express type of transit operation between outlying sections and downtown areas. Bus operation with frequent stops does not appear practical on expressways. Such stops would have to be designed with separate stopping areas and speed change lanes for entering and leaving without interfering with through movements. They logically should be at important cross streets where transfers to crossing transit lines would be desirable. Provisions for separate stops at such locations are costly since they involve widening or lengthening grade separation structures, and the speed change lanes hamper the design and operation of ramps if interchange is also to be provided. In addition, a poor service is rendered patrons when they are required to climb stairs or ramps.

Where interchange ramps are provided, bus operation and stopping problems are solved economically and superior service is given patrons, if the buses use the interchange ramps, stop at turnouts at the level of the cross streets, then use ramps again for returning to the expressway. Where cross traffic is not heavy, buses can cross the intersecting street. Good designs have been made in which buses U-turn on to ramps leading back to the expressway after loading passengers, thus avoiding the crossing of the intersecting street. This is not possible, of course, where single one-way ramps are provided in each quadrant of an intersection. Most bus companies are not partial to operation on expressways where stops necessarily are few and far between, since bus operation is profitable and patrons are served best where bus stops are frequent and near the destinations of patrons.

Mr. Weiner cites the need for better use of existing facilities. This need is obvious for such existing facilities as streets, intersections, and control devices, but the need also exists for better operation of expressways. In large metropolitan areas existing expressways frequently become clogged during peak load hours when freedom of movement is needed most. Sometimes, because of inadequacies of design, such as lack of a shoulder, a stopped vehicle on a through

lane can jam traffic in the same direction on all lanes. Sometimes congestion is due to simple overload. Whatever the cause, slowing down of traffic to 25 miles per hr or less results in use of the expensive facility for volumes less than capacity. For example, an expressway loaded bumper to bumper, with all vehicles stopped, is carrying no traffic in an operational sense. At such times all vehicles would benefit by denying the expressway to vehicles to the amount of the overload. Where jamming occurs regularly, it can be anticipated and police controls arranged for at the critical entering ramps.

Mr. Feld rightly suggests that it is not feasible to design expressways for peak loads. However, his submission of a photograph of the Hendrick Hudson Parkway in New York, N. Y., on Navy Day, although interesting, is academic only. No one would be expected to design for such an extraordinary peak. The American Association of State Highway Officials considers the thirtieth highest hour of the year to be the appropriate hourly volume for use in design, the year to be that for which the facility is being designed. For certain locations the thirtieth highest hour is as little as 8% and for other locations as much as 20% of the average daily traffic.

The writer has not attempted to give closing statements on all matters taken up in the discussions. For some, comments can be found in the paper; others, although worthwhile, are not related to the principles of express highway planning in metropolitan areas. All the discussions, however, have added much to the worth of the paper and evoke the appreciation of the writer.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DESIGN LIVE LOADS IN BUILDINGS

Discussion

BY JOHN W. DUNHAM

JOHN W. DUNHAM,[9] M. ASCE.[9a]—The reasoning of the paper is carried a step forward by Mr. Narver, who compares the resulting overstresses in two cases with the yield point of structural steel. He also adds a note of warning that the proposed reductions must be applied with good judgment. This latter is true of design rules generally.

Mr. Freeman states that the surveys presented in the paper seem to have been limited in scope and that the procedure was not quite correct inasmuch as only one observation was made on any one panel. He questions the source of the maximum overstress of 30% and the reduction rate of 0.08% per sq ft. Mr. Freeman also indicates that the use of a load of 50 lb per sq ft with a corresponding reduction, depending on the number of floors supported, should give safe values. It is assumed that this comment applies to office buildings, and that he has in mind the loads and the reductions recommended by the Building Code Committee.[3,8] These points will be discussed in order.

Any number of surveys will be limited in scope. The writer is not aware of a criterion of sufficiency in this respect, and feels that it is a matter of judgment. More surveys would be helpful but it is doubtful that they would change the picture appreciably.

If the offices in the Equitable Building (surveys of which have been reported[2]) may be assumed to average 500 sq ft in area, the total area surveyed in the five office buildings was 210,290 sq ft. The surveys reported in the paper included over 2,500 panels of office building floors, which had a total area of nearly 430,000 sq ft. There were about 140 columns which appeared to be loaded in such a manner as to be significant.

The arrangement of the load on each of the elements at the time it was surveyed indicates the arrangement that might be found on any of the others

NOTE.—This paper by John W. Dunham was published in April, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1946, by D. Lee Narver, and N. N. Freeman.

[9] Asst. Chf. Structural Engr., Public Bldgs. Administration, Washington, D. C.

[9a] Received February 20, 1947.

[3] "Minimum Live Loads Allowable for Use in Design of Buildings, Report of Building Code Committee," U. S. Dept. of Commerce, November 1, 1924.

[8] "Steel Construction," A.I.S.C., New York, N. Y., 1930, p. 52.

[2] *Engineering News-Record*, March 29, 1923, p. 584.

at another time. The writer feels that the arrangement of loads in space gave a valid indication of what the distribution on any one element would be in time.

Effort was directed toward determining the actual distributions and combinations in a large number of instances so as to obtain an indication of the probable maximum for a given class of element. It was known in advance that more severe loadings were possible. It was desired to find out what does happen, not what might happen.

The probability of overloading an office building with people is remote. The areas in which the very heavy loads occur represent a small percentage of the total areas surveyed and they are not spaces in which people congregate.

Although the suggested reductions well represent the worst conditions found in the surveys and estimated for the apartments, it was recognized that more severe conditions are possible, however remote their probability. It was this consideration that led to the formula which sets a limit on the proposed reductions. In the writer's opinion the possibility of dangerous overstress is thereby practically eliminated.

The proposed reduction of 0.08% per sq ft is empirical and resulted from the writer's study of the survey data. It gives values that start at no reduction for zero area and correspond well with the average live loads found on the columns carrying the heaviest live load at about 800 sq ft of tributary area (see Fig. 3).

The maximum reduction corresponding to 30% overstress was set to keep the overstress conservatively within the elastic limit of steel and well below the ultimate strengths of concrete and wood. It should be noted that nowhere in the buildings surveyed would the overstress have equaled 30%.

Adoption of the reduction rate and the limiting formula by the American Standards Association seems to indicate that qualified structural engineers concur with the writer's judgment on these two points.

In view of the surveys reported in this paper, and elsewhere,[2] it is contended that the use of 50 lb per sq ft as a live load in office buildings amounts to using an initially reduced live load. The report on the Equitable Building indicated a live load of 87 lb per sq ft in one office. The survey of the Veterans Administration Building, which had about the same average load as the buildings reported on previously,[2] indicated a maximum live load of 90 lb per sq ft. The writer believes that the live load of 50 lb per sq ft was established with the knowledge that a very small part of the building might receive higher loads, as indicated in the surveys, but with the feeling that they would not be enough higher to cause serious overstress.

The comparison indicated in Fig. 4(*a*) and the accompanying text supports the writer's opinion that a higher basic live load, together with the proposed system of reduction, will give a better balanced design for a given building than would result from the use of the recommendations of the Building Code Committee.[3]

In conclusion, the writer wishes to express his gratitude to the discussers of the paper for the additional argument and the new viewpoints they have presented.

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DISCUSSIONS

MATRIX ANALYSIS OF CONTINUOUS BEAMS

Discussion

By Rufus Oldenburger, Horace A. Johnson, Lloyd T. Cheney, and Paul W. Norton

Rufus Oldenburger,[12] Esq. [12a]—In a paper [13] by the writer, matrices were introduced to clarify the mathematics of the Cross method of structural analysis[14] because the operations involved in this analysis are linear, and the convergence questions can be expressed simply in terms of matrices. The Cross method is an infinite process whereby fixed joints are relaxed so that the moment distribution for a given structure is obtained from the moment distribution when all joints are fixed.

Mr. Benscoter shows that by matrices one can proceed directly from the fixed joint moments to the final moment distribution. This method, however, involves computation of the inverse of a matrix. Because generally it is exceedingly difficult to obtain the inverse of a matrix directly, the series (Eq. 98) is used, in which $[I]$ is the identity matrix and $[Q]$ is a matrix of the same order. This expansion is suggested by the similar expansion (Eq. 97) for $1/(1 - \epsilon)$ in the case of a number ϵ.

That one can work directly from the initial to the final moment distribution by matrices follows from the fact that the final moments can be obtained from the initial moments by solving linear equations. By "directly" is meant "in a finite number of steps." The importance of Mr. Benscoter's use of matrices lies in the fact that this point of view leads to an infinite process through the use of Eq. 98. It is sometimes simple to obtain successive powers of a matrix $[Q]$, so that if the series converges rapidly, a good approximation to the inverse of $[I] - [Q]$ is obtained. Computation of complicated determinants is thus avoided; but to use Eq. 98 is to employ an approximation method of computation. In fact, it has been shown that each term in that equation corresponds to a standard computation cycle and that each cycle is composed of two steps.

Note.—This paper by Stanley U. Benscoter was published in October, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1947, by I. Oesterblom, and Harris Solman.

[12] Prof. of Math., Illinois Inst. of Tech., Chicago, Ill.

[12a] Received January 10, 1947.

[13] "Convergence of Hardy Cross's Balancing Process," by Rufus Oldenburger, *Journal of Applied Mechanics*, December, 1940, pp. A-166 to A-170.

[14] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE Vol., 96, 1932, pp. 1–10.

"In the first step all joints are balanced and in the second step the carry-over calculation is performed in each span."

The great value of Eq. 98 thus lies in the fact that the terms in the series correspond to the standard accepted procedure for "balancing" moments. It is, however, only one method out of infinitely many for applying the Hardy Cross type of method. The approach in the writer's paper[13] on the subject is general and therefore covers the balancing process corrresponding to Eq. 98. In particular, the convergence theory contained in that paper can be applied to Eq. 98. For reasons of exposition both Mr. Benscoter and the writer have restricted themselves to the case of the continuous beam. If the numbers 1, 2, 3, · · · , n be assigned to the supports along a given beam, the convergence proof of the writer for balancing alternately odd-numbered and even-numbered supports goes over without change to the standard process corresponding to Eq. 98.

In the writer's matrix treatment of continuous beams the vector $[M]$ of the desired moments is computed from the relation:

$$[M] = \lim_{n \to \infty} [A]^n [M'] \qquad (147)$$

for the vector $[M']$ of given moments, provided that the balancing process is a cycle repeated forever. For the elements of the vectors occurring in Eq. 147 the total moments at the supports have been used. This was done for mathematical simplicity. Mr. Benscoter's technique of working with left end moments and right end moments, instead of lumping the moments at each support, has considerable value for understanding the theory of the balancing process, and is needed to determine the final left end moments and right end moments. However, the writer's treatment readily extends to the case where the elements in the vectors of Eq. 147 are the individual unlumped moments. Of course, $[A]$ must be changed accordingly.

Matrices are of unquestionable value in understanding methods for obtaining moment distributions in structures. When it comes to actual computations the situation is different. The writer has still to see a computational process in which the explicit use of matrices gives an economy of effort, although matrices are often of great value in prescribing what computational process should be employed. In any case, it is easier to use Eq. 147 for left end moments and right end moments, obtaining the result by computing powers of a matrix, than to use Eq. 98, where one must perform the additional operation of adding powers after they have been computed.

Mr. Benscoter's paper is an excellent exposition of the matrix point of view in structural analysis, and sheds much light on what moment balancing really is. There is still a need, however, for more specific information on the convergence of the Hardy Cross process, in particular for the case where the moments at supports are not lumped. Above all the problem of the rapidity of convergence should be studied. The designer should know how many steps must be taken to obtain a given accuracy. Because of the importance of the subject a complete physical and mathematical treatment should appear in a book on general

structures, beginning from basic fundamentals. In such a treatment matrices would play a dominant role.

HORACE A. JOHNSON,[15] ASSOC. M. ASCE.[15a]—A service has been done for the engineering profession by the presentation of the subject of matrices and determinants. It has always been a source of wonder to the writer that so few engineers are acquainted with these powerful mathematical tools, or even with the most elementary use of determinants.

However, the author has possibly been overenthusiastic in the use of matrices. For example, Eqs. 36 are expressed as $[A][x] = [k]$ in Eq. 39. Although Eq. 39 is a more succinct expression of Eqs. 36, it conveys less meaning to a person not using matrices frequently, as is the case with most engineers, and it in no way reduces the amount of work required for the numerical solution of Eqs. 36.

Eqs. 36 may be solved easily by use of determinants as follows:

$$x_1 = \frac{\begin{vmatrix} 2 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & -1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{vmatrix}} = \frac{41}{45}; \quad x_2 = \frac{\begin{vmatrix} 3 & 2 & -2 \\ 1 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix}}{45} = -\frac{13}{45};$$

$$x_3 = \frac{\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix}}{45} = \frac{10}{45} = \frac{2}{9}$$

In each of these cases the solution is obtained by substituting the constant terms for the coefficients of the desired unknown in the numerator determinants, the remaining terms being the coefficients of the other unknowns. The denominator determinant in each case is composed of the coefficients of the unknowns. This procedure seems simpler than the solution given by the author.

Solution of simultaneous equations by the use of determinants is a systematic approach to the problem. It does not, however, avoid a large amount of numerical work as the expansion of an nth-order determinant gives $n!$-terms; thus, the expansion of a fifth-order determinant gives 120 terms.

LLOYD T. CHENEY,[16] JUN. ASCE.[16a]—The possible advantages of matrix algebra in various methods of structural analysis are called to the attention of engineers by the author of this commendable paper. The painstaking explanation of elementary operations gives the engineer a condensation of these proc-

[15] Engr. (Civ.), Corps of Engrs., War Dept., Sacramento, Calif.

[15a] Received February 1, 1947.

[16] Asst. Prof. of Civ. Eng., Cornell Univ., Ithaca, N. Y.

[16a] Received February 13, 1947.

esses, which would otherwise be found only after considerable search in one or more mathematics texts. This explanation should serve to demonstrate the relative ease of manipulation of matrix algebra and to reassure the engineer that with the necessary practice it can become a most powerful instrument.

Structural engineers are generally introduced to series of simultaneous equations in applying the classical theorem of three moments or the method of slope deflection.[17] A frequently suggested method of solution is that of arranging the coefficients and constant terms in tabular form.[18] By successive manipulations the series is reduced to a single equation containing a single unknown variable. With the variable thus determined, successive substitutions are made until all variables have been found. For example, the system illustrated in Eqs. 36*a*, 36*b*, and 36*c*, Table 2, has been arranged with the steps numbered and the operations noted so that the table is self-explanatory.

TABLE 2.—FORM FOR SOLUTION OF SIMULTANEOUS EQUATIONS

Step number	Operation	Coefficients			Right side of equation, k
		x_1	x_2	x_3	
(1)	(2)	(3)	(4)	(5)	(6)
1	Eq. 36*a*	3	1	−2	2
2	Eq. 36*b*	1	2	3	1
3	Eq. 36*c*	2	−1	4	3
4	Step 2 ×step 1	6	2	−4	4
5	....	1	2	3	1
6	Step 2 ×step 3	4	−2	8	6
7	Step 4 +step 6	10	0	4	10
8	Step 5 +step 6	5	0	11	7
9	Step 2 ×step 8	10	0	22	14
10	Step 9 −step 7	0	0	18	4

Step 10 of Table 2 gives the value for x_3. Study of the table discloses that within it are the matrices shown by Eq. 37, but that no advantage has been taken of matrix algebra in the solution. This suggests that engineers have been working with matrices but have hesitated to use them as such.

Attention has been called to the fact that other methods yield quicker results for a single condition of load because of the possibility of lengthy computations in evaluating the reciprocal of the stiffness matrix $[K]$. It might be well to emphasize that methods for evaluating more cumbersome determinants are not so frightening as the size of the determinant itself would suggest.

The algebraic sum of the products of the elements of a row and their corresponding minors will give a quick means of evaluating larger determinants. Employing this method with the 4-by-4 determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots (148)$$

[17] *Engineering Studies No. 1*, Univ. of Minnesota, Minneapolis, Minn.

[18] "Structural Theory," by Hale Sutherland and Harry Lake Bowman, John Wiley & Sons, Inc., New York, N. Y., 1935, p. 212.

Still larger determinants may be evaluated by expressing them in terms of the algebraic sum of the products of smaller determinants. Consider the 6-by-6 determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix} = | A | \ldots\ldots\ldots\ldots (149)$$

Making the substitutions—

$$\left.\begin{array}{ll} | B | = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & | C | = \begin{vmatrix} a_{14} & a_{15} & a_{16} \\ a_{24} & a_{25} & a_{26} \\ a_{34} & a_{35} & a_{36} \end{vmatrix} \\ | D | = \begin{vmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{vmatrix} & | E | = \begin{vmatrix} a_{44} & a_{45} & a_{46} \\ a_{54} & a_{55} & a_{56} \\ a_{64} & a_{65} & a_{66} \end{vmatrix} \end{array}\right\} \ldots\ldots (150)$$

—Eq. 149 becomes:

$$| A | = | B | | E | - | C | | D | \ldots\ldots\ldots\ldots (151)$$

By using a combination of these methods, plus manipulations to reduce some of the elements to zero, the evaluations may be accomplished. However, although these computations will be routine, there is no avoiding the fact that they will be time consuming.

The new application of matrix algebra should stimulate interest in this mathematical instrument and result in its wider application to common engineering problems.

Paul W. Norton,[19] M. ASCE.[19a]—Attention has been directed by Mr. Benscoter to interesting possibilities in the use of matrix algebra for the solution of structural problems involving simultaneous equations. By taking advantage of the implement afforded by the matrix notation, the author achieves a noteworthy abbreviation of the process of solution by means of determinants in two important respects: (1) For the several determinant ratios expressing the several unknown quantities, he substitutes a single matrix expression, comprehending in itself the solution for all the unknowns; and (2) the reduction of this expression to the final numerical results involves, not the evaluation of as many determinants as there are unknown quantities, but the evaluation of only one, plus a relatively simple operation on the matrices.

The application of these methods to the determination of moments in rectangular frames, such as those commonly encountered in building construction, is of particular interest to the writer. Some earlier studies of a similar nature, followed by careful reading of this paper, have confirmed him in the conclu-

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[19] Cons. Engr., Boston, Mass.

[19a] Received February 24, 1947.

sions that, as applied to such problems, the formation of the fundamental matrix equation follows even more immediately and obviously from an inspection of the stiffness properties of the frame, and of readily derived functions of its loading, and the indicated numerical operations can be done by processes even less laborious than revealed by the author.

To present clearly a discussion which seems to be justified by this view, and by the hope that its demonstration may contribute something to the usefulness of the author's paper, it will be expedient to show the application of the methods under consideration to the solution of a specific example. The derivation of the determinant and matrix expressions, and the routine of computation will be illustrated, with such explanatory comment as seems useful to the purpose. The adaptability of the method to frames in which occur members having varying moments of inertia, and to those subject to translation of joints, or sidesway, will be indicated without details of analysis or numerical illustration.

Notation.—In addition to the notation of the paper the following symbols are introduced:

M_{pq} = clockwise moment acting at point P on the member (or part of member) PQ.

M_f = Fixed-end moment

$M_{f,pq}$ = a moment M_f at point P; that is, the value of M_{pq} if member PQ were fixed at both ends.

$M'_{f,pq}$ = a modified value of M_f; that is, the value of M_{pq} if member PQ were fixed at point P and freely supported at point Q; for a prismatic member, $M'_{f,pq} = M_{f,pq} - \frac{1}{2} M_{f,qp}$.

$\sum M_{f,p}$ = summation of all values of M_f or $(M_f)'$ for all members intersecting at joint P.

K_{pq} = stiffness coefficient of member PQ at end P; the actual stiffness divided by the constant 4 E. For a prismatic member K_{pq} is the $\frac{I}{L}$-value of member PQ. This is the stiffness coefficient as defined in the familiar form of the slope-deflection equation. Its value is one fourth of that defined by the author, and is preferable because it avoids fractions and simplifies numerical work.

K'_{pq} = a value of K_{pq} modified to account for freedom at end Q. For a prismatic member, $K'_{pq} = \frac{3}{4} \times K_{pq}$.

$\sum K_p$ = the summation of all values of K, or K', for all members intersecting at joint P.

Φ = the angular clockwise rotation of joint P, multiplied by 2 E.

z = a parameter moment; z_{pq} and z_{qp} are parameter moments pertaining to member PQ, the derivation of which (by methods to be described) leads to a determination of moment conditions throughout span PQ.

Determinant of the Frame.—The frame in Fig. 11 is that used by Hardy Cross, M. ASCE, to illustrate the procedure described in his paper, "Analysis

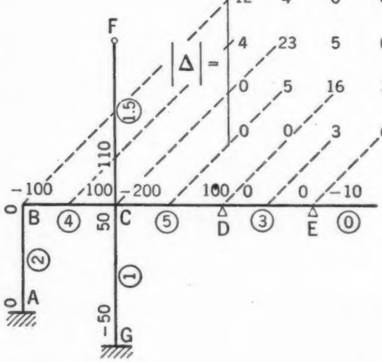
of Continuous Frames by Distributing Fixed-End Moments."[20] The data are here transcribed as given by Professor Cross except: (*a*) The signs of the fixed-end moments are altered to conform to the convention here established; and (*b*) for the member hinged at point F the "modified" stiffness coefficient and fixed-end moment at point C are used. These are:

$$K'_{cf} = \tfrac{3}{4} \times K_{cf} = \tfrac{3}{4} \times 2 = 1.5 \qquad (152a)$$

and

$$M'_{f,cf} = M_{f,cf} - \tfrac{1}{2}\, M_{f,fc} = 80 - \tfrac{1}{2}\,(-60) = 110 \qquad (152b)$$

In Fig. 11, stiffness coefficients are indicated by numerals in circles, and fixed-end moments by unenclosed numerals at ends of members.



$$|\Delta| = \begin{vmatrix} 12 & 4 & 0 & 0 \\ 4 & 23 & 5 & 0 \\ 0 & 5 & 16 & 3 \\ 0 & 0 & 3 & 6 \end{vmatrix} = 12 \begin{vmatrix} \frac{65}{3} & 5 & 0 \\ 5 & 16 & 3 \\ 0 & 3 & 6 \end{vmatrix} = 12 \times \frac{65}{3} \begin{vmatrix} \frac{193}{13} & 3 \\ 3 & 6 \end{vmatrix}$$

$$= 12 \times \frac{65}{3} \times \frac{193}{13} \times \frac{1041}{193} = 20\,820$$

FIG. 11

If the slope-deflection equation were to be used to express the moment at each interior joint in terms of the rotations at that and at adjacent joints, four equations would result relating the four rotations at points B, C, D, and E. One method of solving this set of equations simultaneously involves forming a determinant whose elements are the coefficients of the unknown rotations. This determinant, often called the "determinant of the set of equations," will be referred to herein as $|\Delta|$, the determinant of the frame. It is important in the analysis of the frame, either by determinants or by the author's matrix expressions; it is independent of the loading; and it can be derived by inspection. For its elements, it has functions of the K-values of the members, related to the array of values recorded on the diagram of the frame as illustrated by Fig. 11. The order of $|\Delta|$ is equal to the number of interior joints; the principal diagonal consists of the double sum of the K-values of all members meeting at the corresponding joints; the adjacent diagonals (superdiagonal and subdiagonal) consist of the K-values of the members, in order; and all other elements are zero. Thus, the general expression for the determinant of such a

[20] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, p. 8, Fig. 2.

frame, in which interior joints are denoted by B, C, D, E, is:

$$|\Delta| = \begin{vmatrix} 2\sum K_b & K_{cb} & 0 & 0 \\ K_{bc} & 2\sum K_c & K_{dc} & 0 \\ 0 & K_{cd} & 2\sum K_d & K_{ed} \\ 0 & 0 & K_{de} & 2\sum K_e \end{vmatrix} \quad \text{..........(153)}$$

(it being understood that modified values K' and $(M_f)'$ are to be used in place of K and M_f for members hinged at the remote end).

Rotations of the Joints.—To continue with the analysis of the frame by means of determinants it would be necessary to form another determinant corresponding to each joint, and involving the load terms, or fixed-end moments. These also can be derived by inspection, and the numerical computations can be comparatively brief.

However, there is a marked advantage in being able to express the solution for all parts of the frame in a single matrix equation, as developed by the author. Mr. Benscoter shows that the equivalent of the four expressions for the rotations at the joints B, C, D, and E of the frame, which would result from the determinant analysis, is (letting Δ represent the array of elements in the frame determinant):

$$\begin{pmatrix} \Phi_b \\ \Phi_c \\ \Phi_d \\ \Phi_e \end{pmatrix} = [\Delta]^{-1} \times \begin{pmatrix} -M_{f,b} \\ -M_{f,c} \\ -M_{f,d} \\ -M_{f,e} \end{pmatrix} = \begin{pmatrix} 12 & 4 & 0 & 0 \\ 4 & 23 & 5 & 0 \\ 0 & 5 & 16 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix}^{-1} \times \begin{pmatrix} 100 \\ -60 \\ -100 \\ 10 \end{pmatrix}$$

$$= \frac{1}{20{,}820} \begin{pmatrix} 23\times\frac{343}{23}\times\frac{1{,}851}{343} & -4\times 16\times\frac{87}{16} & 4\times5\times6 & -4\times5\times3 \\ -4\times16\times\frac{87}{16} & 12\times16\times\frac{87}{16} & 12\times5\times6 & -12\times5\times3 \\ 4\times5\times6 & -12\times5\times6 & 12\times\frac{65}{3}\times6 & -12\times\frac{65}{3}\times3 \\ -4\times5\times3 & 12\times5\times3 & -12\times\frac{65}{3}\times3 & 12\times\frac{65}{3}\times\frac{193}{13} \end{pmatrix}$$

$$\times \begin{pmatrix} 100 \\ -60 \\ -100 \\ 10 \end{pmatrix} = \frac{1}{20{,}820} \begin{pmatrix} 1{,}851 & -348 & 120 & -60 \\ -348 & 1{,}044 & -360 & 180 \\ 120 & -360 & 1{,}560 & -780 \\ -60 & 180 & -780 & 3{,}860 \end{pmatrix} \times \begin{pmatrix} 100 \\ -60 \\ -100 \\ 10 \end{pmatrix}$$

$$= \frac{1}{20{,}820} \begin{pmatrix} 185{,}100 & +20{,}880 & -12{,}000 & -600 \\ -34{,}800 & -62{,}640 & +36{,}000 & +1{,}800 \\ 12{,}000 & +21{,}600 & -156{,}000 & -7{,}800 \\ -6{,}000 & -10{,}800 & +78{,}000 & +38{,}600 \end{pmatrix}$$

$$= \frac{1}{20{,}820} \begin{pmatrix} 193{,}380 \\ -59{,}640 \\ -130{,}200 \\ 99{,}800 \end{pmatrix} = \begin{pmatrix} 9.288 \\ -2.864 \\ -6.253 \\ 4.793 \end{pmatrix} \quad \text{..........(154)}$$

—that is, $\Phi_b = 9.288$; $\Phi_c = -2.864$; $\Phi_d = -6.253$; and $\Phi_e = 4.793$.

Before proceeding, attention may be directed to the process of reducing a determinant to a numerical value, such as the 20,820 in Fig. 11, and to the routine of computation in determining the elements of the matrix reciprocal, as illustrated by Eq. 154.

Evaluation of a Determinant.—Numerical evaluation of a determinant by repeated summation of minors, as described in most elementary texts and illustrated by the author in Eqs. 2*a* and 2*b*, is a process that becomes increasingly laborious for those of orders higher than the third. (A determinant of the tenth order, with no zeros among the elements, would require writing and evaluating 2,606,500 minors of all orders.) A much briefer method, illustrated by the numerical work of Fig. 11, and of general application, is the following: In any determinant,

$$|D| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (155)$$

let a "modified minor" of the element a_{11} be formed, by substituting for each element a_{jk} of the minor a value b_{jk} derived by the formula:

$$b_{jk} = a_{jk} - (a_{1k} \times a_{j1}/a_{11}) \qquad (156)$$

This modification of the minor does not change its value, by virtue of the theorem:

> "If the elements of any column (or row) of a determinant be altered by adding to each the corresponding element of another column (or row), multiplied by a constant, the value of the determinant is thereby unchanged."

Then $|D|$ has the form:

$$|D| = \begin{vmatrix} a_{11} & 0 & 0 & \cdots \\ a_{21} & b_{22} & b_{23} & \cdots \\ a_{31} & b_{32} & b_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = a_{11} \begin{vmatrix} b_{22} & b_{23} & \cdots \\ b_{32} & b_{33} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \qquad (157)$$

and the evaluation will have proceeded by reduction of the determinant of the nth order to one of the $(n - 1)$th order. The numerical value of any element b_{jk} is usually ascertainable mentally with small effort. Thus, in $|\Delta|$ of the example considered, the element a_{22} is 23, and the "modified" element b_{22} is $23 - 4 \times 4/12 = 65/3$. The presence of zeros among the elements also greatly facilitates the computation; thus, the element a_{32} in $|\Delta|$ is 5, and $b_{32} = 5 - 0 \times 4/12 = 5$, no modification of the element resulting. Furthermore, the successive reductions in order accomplished by this process result in a continued product of numbers, which, expressed as fractions, invariably lead readily to the final result by cancellations of like figures in numerators and denominators.

Formation of the Reciprocal Matrix.—The author has shown (see Eq. 33) that the reciprocal of $[\Delta]$,

$$[\Delta]^{-1} = \frac{\text{adjoint of } [\Delta]}{|\Delta|} \qquad (158)$$

The value of $|\Delta|$ having already been found, it remains to discover the most direct and rapid way to write the adjoint. The author remarks that in this lies the chief difficulty of the computation; it seems to the writer that the device whereby the matrix reciprocal is expressed in the form of a power series (see heading, "Matrix Reciprocal Expressed by Power Series") is of academic interest, and makes little progress toward facilitating the computation.

In obtaining the results in the problem used as an example it has not been necessary to write down any figures other than those appearing in Eq. 154. The fact that all elements of $[\Delta]$ are zero, except those in the principal and two adjacent diagonals, makes easy the formation of its adjoint.

Let a matrix of this form, and of the nth order, be

$$[M] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \qquad (159)$$

Proceeding to form the adjoint by substituting for each element its cofactor, and to evaluate each cofactor (which is a determinant of the $(n-1)$th order) by the process outlined herein under the heading, "Evaluation of a Determinant," and illustrated in the equation in Fig. 11, a matrix (still of the nth order) is obtained such that each element is the product of $(n-1)$ factors, as follows:

Adjoint of $[M]$

$$= \begin{bmatrix} a_{22}\,a'_{33}\,a''_{44} & -a_{21}\,a_{33}\,a'_{44} & a_{21}\,a_{32}\,a_{44} & -a_{21}\,a_{32}\,a_{43} & \cdots \\ -a_{12}\,a_{33}\,a'_{44} & a_{11}\,a_{33}\,a'_{44} & -a_{11}\,a_{32}\,a_{44} & a_{11}\,a_{32}\,a_{43} & \cdots \\ a_{12}\,a_{23}\,a_{44} & -a_{11}\,a_{23}\,a_{44} & a_{11}\,a'_{22}\,a_{44} & -a_{11}\,a'_{22}\,a_{43} & \cdots \\ -a_{12}\,a_{23}\,a_{34} & a_{11}\,a_{23}\,a_{34} & -a_{11}\,a'_{22}\,a_{34} & a_{11}\,a'_{22}\,a''_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad (160)$$

Observation of easily stated rules to which the elements of the adjoint conform enables one to write it at once for a continuant matrix of any order:

(a) The subscripts of the $(n-1)$ factors of each element of the adjoint are, in order, those of the elements in the principal diagonal of the cofactor of the corresponding element of the matrix.

(b) The sign of the element is the sign of the cofactor.

(c) Primes occur only on factors whose two subscripts are equal, and greater by unity than the two equal subscripts of the factor next preceding. Such a factor immediately following a primed factor is double primed.

(d) The numerical value of a primed factor (as a'_{77}) is a modified value of the corresponding element a_{77} of $[M]$, derived by the formulas:

$$a'_{77} = a_{77} - a_{67} \times a_{76}/a_{66} \qquad (161a)$$

and

$$a''_{77} = a_{77} - a_{67} \times a_{76}/a'_{66} \ldots\ldots\ldots\ldots\ldots\ldots (161b)$$

The application of these principles leads to the matrix shown in Eq. 154 as the adjoint of $[\Delta]$. For example, the cofactor of the element $b_{22} = 23$ of $[\Delta]$ is:

$$\begin{vmatrix} 12 & 0 & 0 \\ 0 & 16 & 3 \\ 0 & 3 & 6 \end{vmatrix} = 12 \times 16 \left(6 - \tfrac{9}{16}\right) = 1{,}044 \ldots\ldots\ldots\ldots (162)$$

—and similarly for the evaluation of the other cofactors. The factors of each element of the adjoint can be written quickly, either by using Eq. 160 as a formula, or directly from an inspection of the matrix. (It helps in concentrating the attention on the diagonal of the cofactor if one obscures the row and column of the matrix which contain the element whose cofactor is being observed.)

Bending Moments in Members of the Frame.—The solution for the joint rotations as shown by Eq. 154 leads easily to graphical determination of end moments in the members of the frame. Two parameters, z_{pq} and z_{qp}, are introduced for each member PQ, as defined in the notation. For example, for member CD, Fig. 11:

$$z_{cd} = K_{cd}\,\Phi_c = 5\,(-\,2.8645) = -\,14.323 \ldots\ldots\ldots\ldots (163a)$$

and

$$z_{dc} = K_{dc}\,\Phi_d = 5\,(-\,6.2536) = -\,31.268 \ldots\ldots\ldots\ldots (163b)$$

These quantities have the character and dimensions of bending moments.

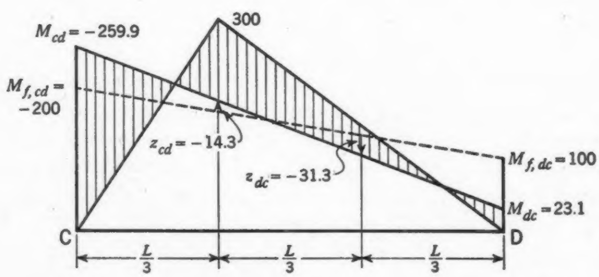


FIG. 12

From points C and D, Fig. 12, ordinates $M_{f,cd}$ and $M_{f,dc}$ (representing the fixed-end moments) are erected, and from the line joining them, at the third points of the span, the values of z_{cd} and z_{dc} are laid off, in the direction indicating negative rotation about points C and D, respectively. The extremities of these lie on the moment line M_{cd}-M_{dc}, whose ordinates at the ends of the span represent the end moments in span CD, under the given conditions of loading and restraint.

The analytical equivalent of the foregoing construction is the expression for the end moments given by the slope-deflection equations:

$$M_{cd} = M_{f,cd} + 2\,z_{cd} + z_{dc} = -\,200 - 28.645 - 31.268 = -\,259.913 \quad (164a)$$

and

$$M_{dc} = M_{f,dc} + 2\,z_{dc} + z_{cd} = 100 - 62.536 - 14.323 = 23.141 \quad (164b)$$

The intercepts of the ordinates between the line M_{cd}-M_{dc} and the moment diagram for member CD simply supported (supplied in Fig. 12 from data not given, but consistent with the fixed-end moments produced) show graphically the bending moment conditions throughout the span CD.

Sidesway.—The occurrence of joint translation, or sidesway, introduces into the problem additional unknown quantities relating to each of the members subject to rotational movement. The order of the matrices and determinants involved in the solution is increased correspondingly. For the ordinary rectangular building frame subject to horizontal displacement of joints, the number of different unknowns so involved is equal to the number of stories; thus for a one-story frame one additional row and column of elements occur in the matrix.

The matrix expression for the solution of this problem is derived without difficulty; however, the fact that the matrix contains elements other than zero in all diagonals, increases the labor of computation considerably, because the derivation of the reciprocal by inspection as illustrated in Eq. 154 is not possible.

Nonprismatic Members.—The method of analysis by matrix computations is applicable also to frames having haunched members. It is necessary, however, to derive modified values of the stiffness factors and load terms relating to the haunched members, and to generalize the fundamental expressions. The author has illustrated the procedure for the case of a continuous beam (see heading, "Numerical Example," and Fig. 5), using values derived from charts published by the Portland Cement Association,[7] and the method of moment distribution.

It is perhaps inappropriate to the purpose of a discussion, the aim of which is to demonstrate the practical usefulness of the matrix analysis in ordinary building design, to pursue the subject into the intricacies of precise results where nonprismatic members are involved. Usually the requirements of design can be satisfied with safety and reasonable economy, if based on a qualitative knowledge of the effects of haunching. However, the writer has found it helpful to his understanding of the principles involved, to derive the generalized values entering into the matrix and slope-deflection equations, in terms of constants made available in tables of the properties of haunched members by Walter Ruppel, M. ASCE.[21] The additional notation is as defined by Mr. Ruppel,[22] except that to avoid conflict I_o is used to represent the minimum moment of inertia of a member; and K_o, to represent the ratio I_o/L.

The stiffness coefficients and fixed-end moments are given by the following formulas:

[7] "Continuous Concrete Bridges," Portland Cement Assn., Chicago, Ill., 1939.

[21] *Transactions*, ASCE, Vol. 90, 1927, p. 152.

[22] *Ibid.*, p. 158.

$$K_{ab} = K_o \times \frac{1 - v}{4\,p\,(1 - u - v)} \qquad K_{ba} = K_o \times \frac{1 - u}{4\,q\,(1 - u - v)} \ldots (165a)$$

$$K'_{ab} = K_o \times \frac{1}{4\,p\,(1 - v)} \qquad K'_{ba} = K_o \times \frac{1}{4\,q\,(1 - u)} \ldots\ldots (165b)$$

$$M_{f,ab} = W\,L \times \frac{t\,u - s\,(1 - v)}{1 - u - v} \qquad M_{f,ba} = W\,L \times \frac{t\,(1 - u) - s\,v}{1 - u - v} \ldots (165c)$$

$$(M_{f,ab})' = -\,W\,L \times \frac{t}{1 - v} \qquad M_{f,ba} = W\,L \times \frac{s}{1 - u} \ldots\ldots\ldots (165d)$$

$$z_{ab} = \frac{K_o}{2\,p} \times \Phi_a \qquad z_{ba} = \frac{K_o}{2\,q} \times \Phi_b \ldots\ldots\ldots\ldots\ldots (165e)$$

Eqs. 165 are general expressions, yielding the values for prismatic beams when the dimensions of the haunch disappear. The matrix expression (corresponding to Eq. 154 for prismatic beams) is:

$$\begin{bmatrix} \Phi_a \\ \Phi_b \\ \Phi_c \\ \Phi_d \\ \cdots \end{bmatrix} = \begin{bmatrix} 2\sum K_a & \frac{2\,u}{1-u} K_{ba} & 0 & 0 & \cdots \\ \frac{2\,v}{1-v} K_{ab} & 2\sum K_b & \frac{2\,u}{1-u} K_{cb} & 0 & \cdots \\ 0 & \frac{2\,v}{1-v} K_{bc} & 2\sum K_c & \frac{2\,u}{1-u} K_{dc} & \cdots \\ 0 & 0 & \frac{2\,v}{1-v} K_{cd} & 2\sum K_d & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}^{-1} \times \begin{bmatrix} -\sum M_{f,a} \\ -\sum M_{f,b} \\ -\sum M_{f,c} \\ -\sum M_{f,d} \\ \cdots \end{bmatrix} \ldots (166)$$

When the joint rotations, Φ, have been ascertained, and the parameters, z, computed by Eq. 165e, the graphical construction for moments may proceed exactly as in Fig. 12, the ordinates z being erected at the u and v points of the span; or the moments may be computed as

$$M_{ab} = M_{f,ab} + \frac{1 - v}{1 - u - v} z_{ab} + \frac{u}{1 - u - v} z_{ba} \ldots\ldots\ldots (167a)$$

and

$$M_{ba} = M_{f,ba} + \frac{1 - u}{1 - u - v} z_{ba} + \frac{v}{1 - u - v} z_{ab} \ldots\ldots\ldots (167b)$$

these being the general forms of Eqs. 164.

Summary.—For a rectangular frame composed of columns and beams, the matrix expression for the joint rotations is derived directly from an inspection of the frame, without the writing of equations. The numerical evaluation of the expression follows an established series of arithmetical operations, which is accomplished easily, with little likelihood of error, and after a little practice, relatively rapidly. The result of the evaluation is translated at once, algebraically, into expressions for end moments in all members, or graphically, into a bending moment diagram for the entire frame.

The matrix solution is a routine of computation rather than a method of analysis. It has the merit of precision, being neither an approximate method nor a method of successive approximations, but an exact method just in so far as the physical data on which the solution is based can be said to be exact. From one point of view (which seems to the writer well grounded) the method shares a defect common to all formula methods. No visualization of the manner of action of the structure attends the solution. The process of computation is mechanical—one feeds the data into the receiving end and, all operations being faithfully performed, extracts the correct result. There is no possibility of applying common sense or judgment to detection of errors at intermediate stages. In this respect the process of approaching results by such means as moment distribution will continue to appeal to most designers as having advantages quite outweighing those of an analysis by equations or formulas.

Nevertheless, Mr. Benscoter's contribution is a valuable one, not only in its application to the analysis of indeterminate structures in the limited field covered by this discussion, but more importantly in promoting acquaintance with an algebraic notation, that is somewhat unfamiliar to many engineers, but which is capable of serving as a powerful tool in the solution of problems involving a number of linearly related unknown quantities.

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DISCUSSIONS

RECHARGE AND DEPLETION OF GROUND-WATER SUPPLIES

Discussion

BY RAPHAEL G. KAZMANN

RAPHAEL G. KAZMANN,[4] Assoc. M. ASCE.[4a]—An excellent summary of the present status of prevailing thought on the permanence and productivity of ground-water supplies is contained in this paper. Unfortunately the paper is too short to do more than summarize various situations. To focus the attention of the profession on the economic implications resulting from just one study of a specific ground-water situation, reference is made to the High Plains of Texas where, in 1943, 400,000 acres of land were irrigated from wells. Under the heading, "Examples of Aquifers with Relatively Small Perennial Supplies," the author states: "It is evident that the rate of pumping will eventually have to be curtailed greatly in this region."

Although the author does not state how long "eventually" may be, it is probable that within the next generation violent changes in the agricultural activity of the area will result from a shortage of ground water. A number of farmers will lose the major part of their capital investments in pumping plants, distribution systems, and farm buildings as a result of ground-water depletion; yet a sound, controlled program of ground-water development would enable a smaller amount of irrigation farming to continue indefinitely in the High Plains area.

In the other direction are the economic implications of the possibilities of increasing the usable ground-water supplies by improved methods of ground-water extraction. Mr. McGuinness states (under the heading, "Examples of Aquifers with High Rates of Recharge and Large Perennial Supplies"):

> "There is much recharge from rainfall and melting snow, but the [Ohio] river forms a major source of recharge at some places where wells have been constructed near the river and are pumped heavily enough to lower the water table adjacent to the stream. Examples of such supplies * * * include the public water supply at Parkersburg, W. Va., and the water supplies

NOTE.—This paper by Charles L. McGuinness was published in September, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1947, by E. A. Vaubel.

[4] Hydrologic Engr., Ranney Method Water Supplies, Inc., Columbus, Ohio.

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at large war plants at Point Pleasant, W. Va., and Charlestown, Ind. At one war plant as much as 50 mgd has been pumped from seven wells in a 2-mile stretch along the banks of the Ohio River. * * * Many large industrial and municipal water supplies have been developed from wells that are so situated as to induce recharge from the streams, and many others undoubtedly will be developed in the future."

Mr. McGuiness did not mention that the "wells," referred to at the war plants, were a relatively new development in ground-water engineering—that is, the radial, horizontal water collectors which are the equivalent of infiltration galleries laid on the bottom of an aquifer. The special virtue of these units is their capacity to operate with utmost efficiency in relatively thin aquifers where conventional wells are impracticable because of the smallness of the available drawdown. Horizontal collectors have their screens projected, horizontally, as close as possible to the bottom of the aquifer. Thus, they operate with equal efficiency as long as a little water remains in the aquifer. In the installations supplied by the infiltration of river water, the potential size of the differential head between the river and the collector or infiltration gallery is a maximum, thus making the supply as large as possible, and permanent as well.

Probably the outstanding installation of horizontal infiltration collectors was made in 1942 at the Wabash River Ordnance Works near Clinton, Ind., where an average production of 72 mgd and a maximum production of almost 89 mgd was obtained from six units.[5]

The value of the additional, permanent ground-water supplies to municipalities and industries is incalculable. However, the horizontal infiltration collector, or any other type of infiltration gallery, although it will develop many new ground-water supplies, is certainly not the last technical advance which will affect the economic availability of ground water. The history of ground-water development shows that improved drilling and pumping methods have increased available ground-water resources. The horizontal water collector too has increased the potential ground-water supply by making thin infiltration aquifers economically available. Finally, a lessening of power costs through the use of new sources of energy would tend to agument available ground-water supplies.

Thus, from an engineering viewpoint, the principal addition needed in Mr. McGuinness' excellent paper is a short section dealing with the effect of technology in increasing the potentially useful quantity of ground water. Applied hydrology must consider engineering techniques.

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[5] "Induced Infiltration Supplies Most Productive Well Field," by Raphael G. Kazmann, *Civil Engineering*, December, 1946, p. 544.

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DISCUSSIONS

STRENGTH OF BEAMS AS DETERMINED BY LATERAL BUCKLING

Discussion

BY H. N. HILL, AND H. D. HUSSEY

H. N. HILL,[33] Assoc. M. ASCE.[33a]—Inadequacy of the ratio of unsupported span length to flange width $\left(\frac{l}{b}\right)$ as a criterion of the stability of a compression flange against lateral buckling has long been recognized. As early as 1924 S. Timoshenko[34] indicated this inadequacy and presented theoretical solutions for the problem of lateral buckling of I-beams under various loading conditions. There remained, however, the task of reducing the complicated theoretical solutions to some simplified form that could be conveniently applied by the designing engineer and yet that would retain a satisfactory degree of accuracy. The significant contribution of the present paper lies in the development of a simple parameter relating the stability of the compression flange to the dimensions of an I-beam, with sufficient accuracy for many desgin purposes.

The price of simplicity in design formulas is generally a restriction in the field of application and the acceptance of some sacrifice in accuracy. There can be no doubt that the simple design formulas proposed by the author are a vast improvement over the usual formulas involving only l/b. Moreover, the author has demonstrated, by comparison with theoretical solutions, that, for a wide group of I-beams, the simple formulas have a degree of accuracy sufficient for many design needs. It may be well, however, to examine the limitations of these simple formulas, which can best be done by examining the steps necessary to derive the simple formulas on a rational basis.

NOTE.—This paper by Karl de Vries was published in September, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1946, by George Winter, and David B. Hall; February, 1947, by Theodore R. Higgins; and March, 1947, by Neil Van Eenam.

[33] Asst. Chf. Engr., Design Div. of Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

[33a] Received February 13, 1947.

[34] "Beams Without Lateral Support," by S. Timoshenko, *Transactions*, ASCE, Vol. LXXXII, 1924, p. 1247.

The buckling stress in the compressive flange of a symmetrical I-beam, with loads applied at the neutral axis, can be expressed as:

$$f_{cr} = R \sqrt{\frac{I_y\, K\, d^2}{I^2_x\, l^2} + T\, \frac{I^2_y\, d^4}{I^2_x\, l^4}} \qquad (75)$$

in which

$$R = \frac{c}{2}\, \frac{E}{\sqrt{2(1 + \mu)}} \qquad (76)$$

$$T = \frac{2(1 + \mu)\, \pi^2}{4} \qquad (77)$$

$$I^2_y = \frac{b^6\, t^2}{36}\, ; \quad I^2_x = U\, \frac{b^2\, t^2\, d^4}{4} \qquad (78)$$

and

$$K = W\, \frac{2\, b\, t^3}{3} \qquad (79)$$

In Eqs. 76 through 79, c is a coefficient dependent on the nature of loading and restraint; μ is Poisson's ratio; U is equal to $\left(1 + \frac{t_w\, d}{6\, b\, t}\right)^2$; W is equal to $1 + \frac{t^3_w\, d}{2\, b\, t^3}$; and t_w is the thickness of the beam or girder web.

By substitution in Eq. 75,

$$f_{cr} = R \sqrt{\frac{4}{9}\, \frac{W}{U}\, \frac{b^2\, t^2}{d^2\, l^2} + \frac{T}{9}\, \frac{1}{U}\, \frac{b^4}{l^4}} \qquad (80)$$

For steel beams, in which $E = 29{,}000{,}000$ and $\mu = 0.28$, $R = 9{,}060{,}000\ c$ and $T = 6.31$. To evaluate U and W, twenty-three I-sections representing extremes in wide flange beams, light beams, joists, and American standard beams have been considered. The limiting values obtained are $0.75 < \frac{W}{U} < 1.07$ and $0.44 < \frac{1}{U} < 0.81$. Eq. 80 can now be written:

$$f_{cr} = 9{,}060{,}000\ c \sqrt{C_1 \left(\frac{b}{l}\right)^2 \left(\frac{t}{d}\right)^2 + C_2 \left(\frac{b}{l}\right)^4} \qquad (81)$$

in which $0.33 < C_1 < 0.48$ and $0.31 < C_2 < 0.57$.

An equation similar in form to Eq. 7 can be obtained by neglecting the second term under the radical in Eq. 81, which then becomes

$$f_{cr} = \frac{9{,}060{,}000\ c\ \sqrt{C_1}}{\frac{l\, d}{b\, t}} \qquad (82)$$

For a uniformly distributed load at the neutral axis and the ends of the unsupported length simply supported against lateral deflection, c is equal to 3.54.[35] Comparing Eq. 82 with the equation for long beams for case (*b*),

[35] "The Lateral Instability of Deep Rectangular Beams," by C. Dumont and H. N. Hill, *Technical Note No. 601*, National Advisory Committee for Aeronautics, Washington, D. C., 1937.

Fig. 4, $32{,}100{,}000 \sqrt{C_1} = 24{,}000{,}000$; or $\sqrt{C_1} = 0.748$. According to the limiting values for C_1 in Eq. 81, for the group of I-sections considered, $\sqrt{C_1}$ may be as low as 0.575. Consequently, for beams having extremely large values of l/b (in which case the second term under the radical in Eq. 81 is truly negligible), the author's equation can give buckling stress values as much as 30% higher than the theoretically correct values. This situation can only obtain in extremely long spans of narrow flange beams, which will buckle at very low stresses, and is very probably of no importance in ordinary design problems. It may be well to keep this limitation in mind, however, in certain handling and erection problems involving long narrow beams.

The inaccuracies introduced by neglecting the second term under the radical of Eq. 81 will depend on the unsupported span length of the beam. Since the second term varies inversely as l^4 and the first term varies inversely as l^2, it is apparent that the importance of the second term increases as the span length decreases. This accounts for the increased discrepancy between the plotted points and the curves (within the region of elastic action) as l decreases in the plots of Figs. 2, 4, and 5. The curves become increasingly conservative as the span length decreases. Again the discrepancies, as indicated by the plots in Figs. 2, 4, and 5, do not constitute a valid objection to the use of the proposed formulas for general design of standard solid section I-beams of structural grade steels. When dealing with high-strength alloy steels and high-strength lightweight alloys of aluminum and magnesium, however, relatively short beams can buckle at stresses within the elastic range. In such cases, theoretical buckling stress values may be more than 50% greater than the values obtained by simplified formulas like Eq. 7. In structures in which these materials are used, lightweight is generally of primary importance, and the introduction of refinements in design is warranted to secure maximum weight saving. Such refinements in design usually entail some sacrifice in simplicity of design formulas.

Possibly the earliest instance of the inclusion of a rational treatment of the lateral buckling problem in a set of design rules is the "Structural Aluminum Handbook,"[36] published in 1938. The treatment is based on a direct application of Eq. 75, for the case of a beam under pure bending, with the ends of the laterally unsupported length simply supported against lateral deflection. The value of c in Eq. 76 for this case is π. (Values of c for numerous other conditions of loading—on the neutral axis—and restraint are available.[35] The fact that the author has been able to represent the cases of flange loading by the same type of formula as that applicable to loading on the neutral axis, and with about the same degree of accuracy, suggests that the cases of flange loading might also be adequately represented by Eq. 75, by substitution of the proper value for c.) Since the equivalent slenderness ratio method is used throughout this handbook to handle buckling in the plastic range, the buckling parameter is expressed in terms of an equivalent radius of gyration ρ:

$$\rho = \sqrt{\frac{0.2}{S_e} \sqrt{I_y (K\, l^2 + 13.1\, I_f\, d^2)}} \quad \ldots\ldots\ldots\ldots\ldots (83)$$

[36] "Structural Aluminum Handbook," Aluminum Co. of America, 1938.

in which S_c is the section modulus about the axis normal to the web (compression side); and I_f is the moment of inertia of the compression flange about an axis through the centroid of the flange, parallel to the web ($I_f = \frac{I_y}{2}$ for symmetrical I-sections—the term I_f being introduced to make the equation applicable to I-sections with unequal flanges).

Although the above equation may have a formidable appearance to a designer accustomed to the utmost simplicity in design formulas, the quantities involved are handbook values, or, in the case of I_f, are easily computed, and the mathematical manipulations involved can be readily and rapidly performed with the aid of a slide rule. When designing lightweight structures in high-strength and low-modulus materials, the appreciable saving in weight that may be realized by the use of the more complicated Eq. 83 will easily outweigh the slight increase in effort required for its application.

The author is to be highly commended for having developed the simple buckling parameter $\frac{l\,d}{d\,t}$ for I-beams and for having proposed a set of simple design formulas which should adequately handle most design problems concerned with the lateral buckling of solid I-beams of structural grade steels. The writer feels, however, that it is desirable that the limitations of design formulas based on this parameter be recognized, and it is important that such formulas shall not be considered as providing adequate design methods for all problems involving the lateral buckling of beams.

H. D. HUSSEY,[37] M. ASCE.[37a]—There has been a need for a formula that takes account of the torsional property of the beam. The basic theory of slender beams has been published for many years, but it has remained for Mr. de Vries to demonstrate how it can be put into simple form.

Eq. 9 appears unduly conservative for use as a general design formula. The author has considered several types of loading but he has omitted any reference to a beam under pure bending, which is the most general form of beam loading. Loadings covered by the author should be considered only as special cases. A formula will be developed, in this discussion, for a beam under pure bending.

When an abstract of this paper was read at the Annual Meeting of the Society in 1945, the following beam formulas were in general use: American Institute of Steel Construction specifications for buildings—

$$f_a = \frac{22{,}500}{1 + \frac{l^2}{1{,}800\, b^2}} \quad \text{(maximum 20,000)} \qquad \text{(84a)}$$

and American Railway Engineering Association and American Association of State Highway Officials specifications for bridges—

$$f_a = 18{,}000 - 5\,\frac{l^2}{b^2} \qquad \text{(84b)}$$

[37] Designing Engr., Am. Bridge Co., New York, N. Y.

[37a] Received February 28, 1947.

in which f_a is the maximum allowable compressive unit stress in bending. When Eq. 9 is written in the form—

$$f_a = \frac{12{,}000{,}000\ t/d}{\frac{l}{b}} \qquad \text{(84c)}$$

—the stresses f_a can be plotted as ordinates against the corresponding ratio l/b as abscissas. Each beam will thus be represented by a separate graph, depending upon the value of the ratio t/d, as shown in Fig. 13. Curves for Eq. 84*c* in Fig. 13 represent the following beam sections: (*a*) 12 B 14 J (t/d = 0.0188); (*b*) 12 B 16½ L (t/d = 0.0224); (*c*) 30WF108 (t/d = 0.0255); and (*d*) 36WF230 (t/d = 0.0351). There are more than forty rolled beams with values of t/d less than 0.0351. These include practically all the popular lightweight sections of each depth. It will be noted that the strength of these forty beams, when calculated by Eq. 84*c*, is considerably less than when calculated by Eq. 84*a*. The strength of many of these beams falls far below that represented by Eq. 84*b*, in spite of the fact that their strength is derived from a basic unit stress of 20,000 and that Eq. 84*b* is based on 18,000.

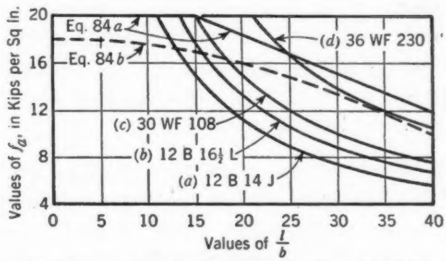

FIG. 13.—ALLOWABLE STRESSES FOR A SIMPLE BEAM, AS COMPUTED BY EQS. 84

For example, the solution of Eq. 84*a*, for an assumed ratio l/b = 30, yields a value of f_a = 15,000. In contrast with this, the corresponding value of f_a for the following beams, when calculated by Eq. 84*c*, is:

Section	f_a
30WF108	10,200
12 B 16½ L	8,960
12 B 14 J	7,520

These examples show that the allowable stresses for many beams, when derived from Eq. 84*a*, were actually at, or near, the critical stresses as determined by Eq. 9.

It is interesting to analyze the loads and moments on typical beams, as is done in the following problems:

Example 1.—The theory of lateral buckling of beams, as illustrated by Fig. 1, is based on a simply supported beam in which the moment diagram varies from zero at the two ends to a maximum value at the center. When the unsupported length of the beam is less than the beam span, the moment diagram for this part of the span is different from that for the entire beam in having moments at the two ends.

Example 2.—Assume that a beam supports two equal concentrated loads symmetrically spaced on the beam, and that the loads divide the beam into three unsupported parts. When the three parts are isolated it is found that

the middle part is under pure bending and each of the two end lengths has a moment diagram which varies from zero at one end to a maximum at the other.

Example 3.—Next assume that the beam in Fig. 1, which supports a single concentrated load at its center, is so long that it is held laterally at the third points, dividing the beam into three unsupported lengths. Assume, also, that the concentrated load is perfectly free to move transversely with the beam as the beam deflects sidewise, as shown in Fig. 1. The middle third of this beam has a maximum moment at its center, but there are moments at the two ends of this part that are two thirds as great as the maximum moment.

Example 4.—If the load on the beam in Example 3 were uniform, instead of concentrated, the moment at the two ends of the middle third would be nearly 90% of the maximum moment. It is assumed, of course, that the uniform load is perfectly free to move transversely with the beam as the beam deflects sidewise. If this beam were divided into more than three parts, the moments at the ends of the middle part would be more than 90% of the maximum moment.

Floor beams in railroad bridges which have symmetrically placed loads, fall in the same class as the beam in Example 2. The middle of the beam, which sustains the greatest stress, is under pure bending. Deck plate girders and stringers in the ordinary railroad bridge are in the class with the beam in Example 4, which has a uniform load. A long deck girder is divided into many panels by the lateral system. If one panel near the center of this girder is isolated, moments will be found at the ends of this panel that are nearly as great as the maximum moment in the panel. The uniform load on the top flange of the girder produces very little change in moment throughout the length of the panel. The principal effect of this top flange load is to contribute to the moments at the two ends of the panel. This girder, therefore, should be considered as a beam under pure bending and not as a beam under top flange loading.

One of the principal considerations in this problem is the nature of the load supported by the beam. To apply this beam theory to a practical problem, the load must be perfectly free to move transversely with the beam as the beam deflects sidewise. If the load is not free to move sidewise, the problem is changed radically and the compression flange is greatly strengthened. For example, if the load in Fig. 1 is applied at the top flange of the beam and the load is perfectly free to move sidewise, the strength of the beam is proportional to the values of the factor k as given in Col. 2, Table 1. However, if this load is not free to move sidewise, the problem is transformed into the problem illustrated in Fig. 8, and the strength of the beam is proportional to the factor k as given in Col. 8, Table 1. The strength of the latter beam is much greater than that of the former.

The number of beams in actual structures, which support loads that are perfectly free to move sidewise, is so small that such beams should be considered as special cases. A trolley beam, for instance, which supports a free concentrated load at the bottom flange, is 50% to 150% stronger than the beam used by the author in developing his proposed Eq. 9.

It is apparent from Eq. 4 that there are three fundamental variables in the solution of the beam problem—length, section, and type of loading. In attempting to present a simple formula the author has included only the first two of these variables in the parameter, $l\,d/(b\,t)$, of Eq. 9. He has covered the third variable, the type of loading, by using a single constant—12,000,000—which produces a formula that is safe for the most severe loading, irrespective of the frequency of its occurrence.

S. Timoshenko[38] has demonstrated the theory of beams subjected to pure bending. The value of the critical moment for this type of loading may be expressed as follows:

$$M_{cr} = \frac{\pi^2 E\, I_y}{l^2}\frac{d}{2}\sqrt{1 + \frac{K\, l^2}{6.4\, I_y\, d^2}} \text{ (assuming } \nu = 0.30) \qquad (85)$$

in which ν is Poisson's ratio. The value of the critical buckling unit stress (for a symmetrical section) is:

$$f = \frac{\pi^2 E}{4\left(\dfrac{l\, r_x}{r_y\, d}\right)^2}\sqrt{1 + \frac{K\, l^2}{6.4\, I_y\, d^2}} \qquad (86)$$

in which r_x and r_y are the radii of gyration about the x-axis and y-axis, respectively.

Eq. 86 may be simplified by the use of two close approximations. First, the ratio $\dfrac{d}{r_x}$ for rolled beam sections is nearly constant and equal to 2.45. Second, the torsion constant K can be represented for rolled beam sections by the approximate formula:

$$K = 0.30\, A_s\, t^2 \qquad (87)$$

in which A_s is the sectional area of the beam and t is the thickness of the flanges. (Eq. 87 is exact for many sections. Since K is under the radical in Eq. 86, any error due to the use of Eq. 87 has only a small effect on the value of the critical stress.)

Making use of Eq. 87 and assuming that $\left(\dfrac{d}{r_x}\right)^2 = 6$, Eq. 86 becomes:

$$f = \frac{444{,}000{,}000}{\left(\dfrac{l}{r_y}\right)^2}\sqrt{1 + \frac{1}{22}\left(\frac{l\, t}{r_y\, d}\right)^2} \qquad (88a)$$

The critical buckling stress for rolled beams may be written:

$$f = \frac{444{,}000{,}000}{\left(n\dfrac{l}{r_y}\right)^2} \qquad (88b)$$

[38] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 259.

in which

$$n^2 = \frac{1}{\sqrt{1 + \frac{1}{22}\left(\frac{l\,t}{r_y\,d}\right)^2}} \quad \ldots\ldots (89)$$

Numerical values of n are as follows:

$\frac{l\,t}{r_y\,d}$	n	$\frac{l\,t}{r_y\,d}$	n
0	1.00	8	0.71
1	0.99	10	0.65
2	0.96	12	0.60
3	0.92	14	0.56
4	0.87	16	0.53
5	0.83	18	0.50
6	0.79	20	0.48
		24	0.44

A formula for working stresses can be obtained by dividing Eq. 88*b* by an appropriate factor of safety. Using a factor of safety of 1.65, for instance, the working formula becomes:

$$f_a = \frac{270{,}000{,}000}{\left(n\,\frac{l}{r_y}\right)^2} \quad \ldots\ldots (90a)$$

Eq. 90*a* may be expressed in a very simple form as follows:

$$f_a = \left[\frac{16{,}400}{n\,\frac{l}{r_y}}\right]^2 \quad \ldots\ldots (90b)$$

Eq. 90*b* avoids the use of large numbers that are involved in the solution of Eq. 90*a*.

By Eq. 90*b*, with $l/b = 30$, $l/r_y = 153$, $\frac{l\,t}{r_y\,d} = 3.9$, and $n = 0.875$, the allowable unit stress for the 30WF108 beam is 15,000. This is in sharp contrast with the value of 10,200 determined by Eq. 9 and agrees exactly with the value of 15,000 obtained by the use of Eq. 84*a*. Values of the coefficient k in Eq. 4, for a beam subject to pure bending, are as follows:

$\frac{l}{a}$	k	$\frac{l}{a}$	k
2	2.926	8	6.753
3	3.413	9	7.490
4	3.996	10	8.236
5	4.640	11	8.988
6	5.322	12	9.746
7	6.029	13	10.51
		14	11.27

A comparison of these values of k with those given in Table 1 reveals that a beam supporting a uniform load at the centroid of the section is 13% stronger,

and when supporting a concentrated load at its centroid it is 36% stronger, than it is under pure bending.

The author mentions the erection problem of picking long beams at the ends, as a condition of a uniform load at the centroid. It will be recognized that a long beam hanging in erection slings is not the same problem as the beam shown in Fig. 1. The ends of the beam in Fig. 1 are held in a vertical position against rotation. When a beam is picked off the ground, however, the ends can rotate as the beam buckles laterally. It is hoped that the author will extend his investigation to cover this particular condition.

This problem is of such interest to erection engineers that it deserves further study. One can compute the span length at which a simple-span beam will buckle from its own weight, when the ends are blocked so that they cannot rotate. Using Eq. 85, and increasing it by 13% to make it applicable to a uniform load, will give the critical moment at which such a beam will buckle. Substituting $\frac{w\,l^2}{8}$ for the moment produces an equation from which the desired length can be determined.

When solving Eq. 85 for long beams, the numerical value of the second term under the radical sign becomes so great as compared to unity, that the latter may be neglected. A solution of Eq. 85 then becomes:

$$M_{cr} = \frac{\pi^2 E\, I_y}{5\, l} \sqrt{\frac{K}{I_y}} \qquad (91)$$

Increasing Eq. 91 by 13%, and substituting $M_{cr} = \frac{w\,l^2}{8}$ and $K = 0.30\, A_s\, t^2$, leads to the following equation for rolled beams:

$$l^3 = 9.77\, \frac{E\, A_s\, t\, r_y}{w} \qquad (92a)$$

in which w equals the weight of the beam per inch. Noting that, for steel beams, $w/A_s = 0.2833$ lb (the weight of 1 cu in. of steel), the following simple equation can be written:

$$l = 1{,}016\, \sqrt[3]{r_y\, t} \text{ (inches)} \qquad (92b)$$

Eq. 92*b* gives the length of a simple span rolled beam which will buckle from its own weight. For example, $l = 110$ ft for a 36WF150, and 98 ft for a 30WF108 beam.

If the foregoing simple-span beam is picked up at the ends, the web blocking has been removed and the ends can rotate as the beam deflects laterally. While hanging in the slings, the centers of the two supports will lie in the vertical plane which passes through the center of gravity of the beam. If the twisting of the beam is neglected (assuming an infinite torsional rigidity) and lateral deflection only is considered, the following formula can be derived:

$$l^4 = 120\, \frac{E\, I_y\, h}{w} \qquad (92c)$$

in which h is the distance from the support to the centroid of the beam, in the plane of the web. If the slings are attached to the top flange of the beam, h may be assumed equal to $d/2$. For this case Eq. 92*c* becomes:

$$l^4 = 60 \frac{E I_y d}{w} \qquad \text{(92d)}$$

Introducing the value of $w/A_s = 0.2833$ for steel beams, Eq. 92*d* reduces to:

$$l = 282 \sqrt[4]{r_y^2 d} \text{ (inches)} \qquad \text{(92e)}$$

Solving Eq. 92*e* for the beams considered in connection with Eq. 92*b* a length of 89 ft is determined for a 36WF150 beam, and 79 ft for a 30WF108 beam.

Eq. 92*e* gives the length at which a steel beam will buckle when hanging in slings at the ends, assuming that the torsional rigidity of the beam is infinite. If the true torsional rigidity of the beam were considered, the correct length should be less than that found by Eq. 92*e*.

Acknowledgment.—The writer wishes to acknowledge the assistance given to him by George E. Howe and C. W. Wixom, Members, ASCE, in the preparation of this discussion. The former was the first to indicate the extremely low unit stresses permitted by Eq. 9 for L-beams and J-beams, as compared with those derived from Eq. 84*a*. The latter suggested the form of Eq. 90*b*.